Pre-K to Grade 2 Goals and Standards:
Achieving 21st Century Mastery for All

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If the U.S. is to attain its goal of strong competitive performance in international comparisons, we must restructure the goals and standards for Pre-K through Grade 2 mathematics. This paper summarizes central issues involved in doing this so that we can attain mastery by all children at the end of Grade 2. This paper is drawn from the extensive research literature and from my experiences summarizing this literature for the NCTM research paper on number and operations, from my service on the National Academy of Sciences committee on mathematics learning, and from my extensive curricular research experience over 8 years in urban and suburban classrooms seeking to actualize a research-based world-class mastery curriculum for Kindergarten through Grade 3. More extensive discussion of the research literature in this domain can be found in Fuson (in press, 1992a, 1992b).

The research evidence suggests that it is possible for all children to leave Grade 2 with higher levels of understanding and skill than are presently demonstrated by children in affluent areas. It is vital that we increase the mastery by all through Grade 2 because at present many children leave Grade 2 already so far behind that it is difficult to catch up. Some of these children are in every school, but disproportionate numbers of such children are in schools of poverty. Changing the present situation will enable our country to have many more students successful at upper grades because they will have a strong foundation in the early grades.

**The Need to Focus on Big Coherent Conceptual Grade-Level Chunks**

Several current aspects of U.S. mathematics education coalesce to require that U.S. mathematics curricula become reorganized into big coherent conceptual grade-level chunks that are integrated over grade levels. These aspects are sketched in Figure 1 and discussed below.

We first need to acknowledge that the goal of mastery for all, even in the area of computational fluency, has been an elusive goal at least since the 1950s. It is not the case that the U.S. has had a successful computational curriculum that is now at risk of being thrown over by “math reform.” National reports, national research studies, and international studies have for decades identified many aspects of computation in which results were disappointing. These results were sometimes overshadowed by even worse results for problem solving or applications of calculations, making calculation seem less of an issue than it has consistently been. Many of the calls for math reform focused on understanding have been at least partially focused on teaching for understanding as a way to eliminate computational errors and thus increase computational performances. For example, on standardized tests national Grade 2 norms for 2-digit subtraction requiring borrowing (e.g., 62 - 48) are 38% correct. Many children subtract the smaller from the larger number in each column to get 26 as the answer to 62 - 48. This top-from-bottom error is largely eliminated when children learn to subtract with understanding (e.g., Fuson & Briars, 1990; Fuson, et al., 1997; Hiebert et al., 1997). Building on a foundation of understanding can help all students achieve computational fluency.
Problems with present U.S. math curricula

In comparisons with the curricula of countries achieving well on international comparisons, the U.S. curriculum has been characterized as “underachieving” and as “a mile wide and an inch deep” (McKnight, et al., 1989; McKnight & Schmidt, 1998; Peak, 1996). Successful countries select vital grade level topics and devote enough time so that students can gain initial understandings and mastery of those topics. They do not engage in repetitive review of those topics in the next year; they move on to new topics. In the United States no teacher and no grade level is responsible for a given topic. Topics such as multidigit computations are distributed over several years, doing one digit larger each year. Large amounts of time are devoted at the beginning of each year and of each new topic to teach what was not learned or was learned incorrectly in the year before.

This wastes huge amounts of learning time and bores the students who have mastered concepts. It is also counterproductive because it is much easier to help students build initial correct computational methods than to correct errors. For example, second graders using base-ten blocks for initial learning of multidigit addition and subtraction explained answers and achieved high levels of accuracy that were maintained over time (Fuson, 1986a; Fuson & Briars, 1990). Older students who had been making subtraction errors for years did learn in one session with base-ten blocks to correct their errors, but many later fell back to their old errors (Resnick & Omanson, 1987). Carefully designed practice, help during learning, and other aspects described below are important for computational fluency. But the most crucial necessity at this point is helping students learn in a timely fashion one correct generalizable method that they understand. Such initial learning must be deep and accurate. Only with understanding can interference from later similar notations and methods be reduced.

Most people think that we do not have a national curriculum in the United States. At one level this is true--each of the 50 states has its own grade-level math learning goals, and each district within a state may (and usually does) further specify goals. These goals vary hugely across different states.

The NCTM Standards (2000) also is not a national curriculum. It specifies only loosely defined topics within wide grade bands. Instead, it rather articulates perspectives of balancing across mathematical topics and across goals such as problem solving, communication, and computational fluency.

However, we do have a functional national curriculum, at least at the Kindergarten to Grade 8 level. That curriculum is the topics that are in each grade in the major commercial textbooks published by the for-profit sector. An examination across these textbooks reveals remarkable similarity. This similarity results from the necessity to sell textbooks across all of the different states, with their differing state goals. Even if a company focuses in on the most
populous states, where it is possible to sell the most textbooks, the total topics across these states results in a large list of topics for the grade level. Even though the books are very large, the number of topics means that very few days can be allocated to each topic. There is not time to develop understanding deeply. Because most teachers use a commercial for-profit textbook, most student thus experience these cursory treatments of most topics.

Furthermore, this national curriculum is heavily influenced by large states that require textbooks to go through an approval process to appear on a state-approved list. Some of the largest of these states have large numbers of non-English-speaking students and students living in poverty. This results in reducing expectations of reading and complexity because of assumptions about what is possible for such students to learn. To sell in these states, the textbooks are reduced in complexity, and the learning potential of all students in the nation is thereby reduced.

There are no requirements that commercial textbooks have data about the success of their programs. Because of the rapidly-changing nature of state goals, textbook companies are under very difficult time lines to produce learning materials. Recent take-overs of many textbooks companies and present economic conditions result in much textbook writing being done by short-term writers or development houses rather than in-house writers, further reducing coherence of the learning materials. The physical appearance and the availability of many peripheral materials are often the bases of a choice of a textbook rather than the quality of the learning experiences it provides.

**Equity issues**

Much research indicates that children from diverse backgrounds can learn mathematics if it is organized into big coherent chunks and if children have opportunity and time to understand each domain deeply. A review of many research studies by Kameenui and Carnine (1998) identified “structuring curriculum around big ideas” as a crucial aspect of learning materials. Another large review of studies (Dixon et al., 1998) found a 3-phase instructional method (discussed more below) superior to traditional instruction. This 3-phase model enabled students to learn deeply over a sustained period in one domain. Children from diverse backgrounds can learn mathematics. But they need sustained and supported learning time in connected areas. Learning materials do not need to be “dumbed down” for learners from diverse backgrounds, but they need to be reorganized and focused.

**New learning goals for the 21st century**

Technology and the information age have created new learning goals for mathematics all over the world. Now adults must understand computational methods and be able to use them in a range of situations rather than just get correct answers. Understanding takes time to develop. So these new goals also demand big coherent conceptual chunks.
These new learning goals are accessible to high-poverty classrooms as well as to more affluent classrooms. Knapp and associates (Knapp, 1995; Zucker, 1995) found that successful teachers in high-poverty classrooms supported conceptual understanding by focusing students on alternative solution methods (not just on answers), elicited thinking and discussion about solution methods, used multiple representations and real-life situations to facilitate meaning-making, and modeled ways to probe meaning of mathematical problems or methods. We used similar approaches to obtain above-grade level understanding with multidigit computation in high-poverty Latino classrooms (Fuson, Smith, & Lo Cicero, 1997).

Organizing U.S curricula into big coherent conceptual chunks

There is sufficient research information about what is accessible to students, especially at the Pre-K to Grade 2 levels, to outline ambitious but attainable mastery goals for all children at each of these levels. These goals would focus on the goals most appropriate and most central for these grade levels. They could result in mastery that is beyond the present levels of achievement even in schools with few students of poverty. If states would work together to agree on some core of goals (say 80% of the goals for each level), then several different high-quality programs could be developed to focus on these goals. States could vary the remaining 20% of goals if agreement could not be reached beyond the 80%. Such research-based ambitious goals would provide all states with a solid foundation of successful children at the lower grades who would not be permanently out of the mathematics learning pipeline.

These mastery goals would be grade level specific, not by grade-band (e.g., grades 3-5). Grade-band goals such as used by NCTM and by states can lead to excessive review and non-mastery at any grade level because the grade-band specification is taken to mean “do at each grade 3, 4, and 5” rather than to mean “do well somewhere within the three grades 3, 4, and 5.”

These goals would also not limit what children would or could learn. They would simply be the baseline goals for which schools would be held accountable. Research indicates that U.S. children can learn more than they do at present. Increasing learning in the early grades will make it easier to be more ambitious at the upper grades where U.S. students are considerably behind.

Achieving mastery for all:

The necessity to move from “standardized” to mastery testing

U.S. children’s learning has a scattershot “mile wide inch deep” nature for another reason. Many school districts give standardized tests to assess how students are achieving. Many people think that a “standardized” test is objective, curriculum-fair, contains mostly grade-level items, was developed by experts in a domain, reflects important learning goals in a balanced way, and assesses what students know about the contents for that grade level. Most
standardized tests have none of these characteristics. Profit-making companies develop them with no control over their content other than what sells. Most major textbook companies also have some other part of their company that produces a standardized test (with a different name); this test may be related to the textbook so that it covers topics in that textbook. But tests are quite variable in the topics they cover and in the proportion of topics that are important at each grade level. For example, two prominent tests vary by 2 years in the year at which they assess single-digit multiplication facts (in grade 2 versus grade 4).

The goal of standardized tests is to determine which half of the students are above average and which half are below average (and which schools and districts are above average or below average). A mastery test consisting of representative grade-level items would result in high proportions of students getting high scores. To rank-order students, many items must be difficult or complex. They must use difficult vocabulary or have 2-steps or have difficult formats or be complex in other ways. Otherwise most students would get them right, and the tests could not scale students.

Standardized tests are secret. Teachers, parents, and students are not allowed to know what is on the test. This is supposedly to keep the test “objective.” But in mathematics, it is very easy to make parallel items that are similar. The secrecy really just results in standardized tests not being curriculum-fair. Teachers do not even know what to teach, so students may fail items because they did not have an opportunity to learn those items.

All of these aspects of standardized tests render it difficult to prepare for them sensibly. Frenzied test-preparation time reviewing and practicing many different topics further reduces the chance for sustained learning time on central topics which could enable students to answer questions that are complex rather than simple.

A further problematic aspect of standardized tests is the nature and timing of their results. They do not give sufficient information about performance on particular items so that those can be remediated for individual students (or such information is very expensive). Results often are not returned until the following year or at the very end of the school year, so results cannot be used for instruction for the class in general or for specific students.

Many states have begun to shift to mastery testing. But these are usually at grade 2 or grade 3 at the earliest, where many children may already be too far behind. Districts meanwhile continue to use at many grade levels standardized tests that have all of the above problems. The combination of standardized tests and state mastery tests is overwhelming for teachers, who must prepare for 2 kinds of high-stakes tests about which they necessarily have inadequate information.

Research-based grade-level tests that were public would enable a whole community to work together to help all children in a school achieve mastery on those goals. In the early years,
Achieving 21st Century Mastery for All

this must include some interview components because much early mathematical competence depends on counting skills and on whether children understand and can explain their methods. Periodic mastery testing throughout the year, including some interviewing, would enable teachers to use the testing information for instruction and for obtaining extra help for those students who need it. The money now used for standardized testing could be used to organize such extra helping for students in various ways, including family out-reach programs to help families learn how to help their children in mathematics.

Mastery for all will require additional learning time for some children who enter preschool or kindergarten with less mathematical experience than other children. There are many ways in which such extra learning time might be organized. It might even take as much time as an extra year early in schooling for some children who enter way behind and have no pre-kindergarten experience. In Russia, this is acknowledged by organizing some schools and some classes within schools to take four years to do the content of the first three years of school. Children learn the same content—they just have more time to do so. This is as contrasted to the U.S., where we often pass children with weak backgrounds along and do not allow them to learn the same content as children who enter with stronger backgrounds. They get a “dumbed down” curriculum.

We must not confuse educatedness (present school-relevant knowledge) with educatability (how much a child can learn). At present we often confound these by lowering the goals for those who enter with less school knowledge (low educatedness) by assuming that they are less educatable. Mastery testing, extra help and learning time for those who need it, high ambitious goals for all, reorganized curricula, and teaching as discussed below can greatly increase the amount of mathematics learned by U.S. children.

**Instruction That Facilitates Mastery by All**

**Helpful instructional phases**

What features of classrooms can contribute to mastery by all? A recent review of the literature contrasts the many studies that found an experimental instructional method superior to a traditional control method (Dixon et al., 1998). The less-effective traditional methods involved two phases: A teacher presentation of some topic (with students observing passively) followed by independent student practice of that topic, with or without teacher monitoring, giving feedback, etc.

Superior learning was achieved by effective methods that had 3 phases. In the initial orienting phase, teachers initially involved students in the introduction of the topic through explanations, questions, and discussion: students were active learners whose initial knowledge about a domain was elicited. In the second supporting learning phase, students are helped during a long period to move from teacher-regulated to self-regulated solving. Teachers structured a
significant period of help that was gradually phased out. This helping was accomplished in
different ways: by scaffolded problems and visual or other supports, by peers, and by the
teacher or aides. During this sustained helping period, students received feedback on their
performance, got corrective help so that they did not practice errors, and received (and often
gave) explanations. The third phase of effective instruction focused on long-term remembering.
This involved a brief assessment of students’ ability to apply knowledge to untaught problems
(near transfer) in which students worked independently. Such independent work is best
distributed over time. Distributed practice has been found to be helpful to remembering in a
wide range of studies, as has experiencing similar but related problems and situations. <<Karen,
I worry that many teachers who use textbooks and do the “introductory” ideas will believe they
are doing what you think best. I believe that’s not what you mean, but it’s easy to reduce the
above to that traditional notion. Also, I don’t see you criticizing the really problematic aspects
of Dixon and Carnine’s “review”...but that’s your decision.>>

Other relevant results from reviewed studies were that strategy instruction of various
kinds was superior to not giving such instruction, working fewer problems in depth was more
effective than working more problems quickly, writing as well as solving problems was
helpful, and solving concept examples sequenced to facilitate generalization and discrimination
was helpful.

The implication of all of these results is that all students had sustained supported time to
learn a given domain deeply and accurately. Such deep sustained accurate learning over time is
necessary for complex domains requiring multistep solution methods. Students need to learn the
central principles of a domain (e.g., in multidigit addition and subtraction, that you add or
subtract like multiunits), learn the overall shape of a given method, learn in detail the steps of
the method, and weave this developing knowledge together so that it operates fluidly and
accurately. This is true whether the students invent the method or learn it from other students
or from the teacher. Practice was important, but effective practice was supported by
monitoring and help focused on doing and on understanding. In contrast, “drill and practice”
frequently carries the connotation of rote practice, has little sense of monitoring or feedback,
and no sense of helping or of visual, conceptual, psychological, or motivational support for
learning throughout the practicing phase.

A textbook issue that at present interferes with the more effective three-phase method
(and interferes with effective teacher presentation of topics even in the less-effective 2-phase
traditional approach) is the common misuse of art (i.e., photographs, drawings, cartoons, etc.)
in U.S. math textbooks. In many other countries the art is designed to support conceptual
thinking. In the United States, art frequently distracts from conceptual understanding because it
is irrelevant or overwhelmingly busy. All visual aspects of learning materials need to support learning, not interfere with it.

**Helping diverse learners**

A related review of literature concerning school success of diverse learners (Kameenui & Carnine, 1998) identified six crucial aspects of successful teaching and of learning materials for diverse learners: structuring around big ideas, teaching conspicuous strategies, priming background knowledge, using mediated scaffolding (e.g., peer tutoring, giving feedback about thinking, providing visual supports that provide cues for correct methods), using strategic integration (integration into complex applications to provide distributed practice in more complex situations), and designing judicious review. Diverse learners are those who may experience difficulties in learning because of low-income backgrounds, speaking English as a second language (or not at all), or other background reasons.

Structuring around big ideas has been discussed above. The next three aspects specify aspects of the initial active learning phase and the helping phase in the 3-phase effective teaching model outlined above. Using strategic integration and designing judicious review are aspects of computational fluency that follow deep and effective initial learning in a domain. Strategic integration of various computational methods into moderately complex problems increases problem-solving competence by increasing the range of situations in which students use that computational method. It also provides for distributed practice of the method, one of the most effective kinds of practice.

Judicious review is defined as being plentiful, distributed, cumulative, and appropriately varied. It follows initial deep learning. Distributed and monitored practice requires working one or two examples occasionally, with immediate help for wrong answers. This is important even after successful meaningful learning because the nonsupportive or misleading mathematical words or notations in many domains continually suggest wrong methods (e.g., aligning a 1-digit and a 2-digit number on the left because you write from left to right). Furthermore, many computational domains are similar, and learning new domains creates interference with old domains (e.g., you do **multiply** the tops and bottoms of fractions but you do not add or subtract them if the bottoms are different). Therefore, after deep and successful initial learning, distributed practice of a couple of problems of a given kind can check whether errors are creeping in. Frequently, helping students correct their methods is as simple as suggesting their remembering of original supports. For example, as some errors crept into multidigit methods learned with base-ten blocks, asking students to “think about the blocks” was sufficient for them to self-correct their errors in subtracting with zeroes in the top number (Fuson, 1986a).
The research of Knapp and associates (Knapp, 1995; Zucker, 1995) on attributes of successful high-poverty classrooms underscores these results. They found that a balance between conceptual understanding and skill practice resulted in higher computational and problem solving results by lower-achieving and higher-achieving students. These teachers also provided a “healthy dose” of skill practice. <<It’s unclear what the “also” adds; you already said there was a “balance”…?>>

**Individual differences**

As in other subject-matter areas, substantial social-class and ethnicity differences exist in mathematics achievement (e.g., Ginsburg & Russell, 1981; Secada, 1992). Kerkman and Siegler (1993) found that low-income children had less practice in solving problems and they executed strategies less well. Strategy instruction and monitored practice were therefore recommended for such students. Individual differences as early as first grade cut across gender and income-levels to differentiate children into what Siegler has termed good students, not-so-good students, and perfectionists (Siegler, 1988). Roughly half of the not-so-good students went on to be identified as having mathematical disabilities by fourth grade versus none of the other groups. On single-digit addition tasks, these students were characterized by use of more primitive methods and by more production of errors on problems on which they could have used (but did not use) more accurate but effortful strategies (e.g., counting with their fingers). These students were producing incorrect answers more often, thereby creating responses that competed with their experiences of correct answers. Perfectionists and good students had similar long-term outcomes, but the perfectionists were much more likely to use slower and effortful methods even on simpler problems than were the good students.

These results emphasize that methods of practice should facilitate individuals understanding their own growth and progress rather than the comparing of individuals. Practice should also be varied so that sometimes speed is important but, at other times, use of a method in a complex situation is important. An overemphasis on either could lead to rigidity rather than computational fluency. Not-so-good students need help to learn and use more accurate strategies.

There has been less work on mathematics disability than on reading disability, especially with younger children. Different kinds of mathematics disability have been identified. Geary’s (1994) review identifies four types and recommends different kinds of learning supports for each kind. His results and recommendations are quite consistent with the above research. They emphasize using visual conceptual supports with extra conceptual cues, teaching more-advanced solution methods, drawing the problem situation, and using accessible algorithms that are easy to learn conceptually and carry out procedurally.
Students with semantic memory disabilities have difficulty with verbal, and especially phonetic, memory but many have normal visuospatial skills. These students have great difficulty memorizing basic computations because these rely on a phonetic code. Therefore instructional supports that use visual rather than phonetic cues and teaching strategies for basic calculations are recommended for these students.

Students with procedural deficits use less-advanced methods than their peers. Though many eventually catch up, this long period of using primitive methods may be detrimental. Such children do not seem to invent more-advanced methods as readily as do their peers. Therefore, conceptually-based strategy instruction that helps them use and understand more-advanced strategies such as counting on can be helpful.

Students with visuospatial disabilities have difficulties with concepts that use spatial representations, such as place value. Research is not clear about the developmental prognosis of such children, but suggested methods of remediation are to support visual processing with extra cues. Because directionality is a special problem with such students, the accessible methods described later in this chapter that can be carried out in either direction might be especially helpful for such students.

Difficulties with mathematical problem solving that go beyond arithmetic deficits also characterize some students. Supports for problem solving such as drawing the problem situation that were discussed in the earlier part of this chapter are suggested as useful for these students. Technology may also help provide complex problem-solving situations that are nevertheless accessible to students with math disabilities (Goldman et al., in press).

Teaching to Prepare for Topics in the Upper Grades

Teaching and learning in the lower grades can lay an adequate foundation for later work. Or it can make such work more difficult by leading students to make generalizations that are not accurate. For example, at the present time students make many errors in decimal and ordinary fractions because they incorrectly generalize concepts from whole numbers (e.g., Hiebert & Wearne, 1986; Resnick et al., 1989). Approaching all domains with a focus on the meanings for the notations, and with explicit consideration of what does and what does not generalize, could improve student competence in these advanced domains. Deep understanding of place value and of the regular ten-for-one trades to the left as numbers get larger can facilitate understanding decimal fractions as regular one-for-ten trades to the right, as quantities get smaller. Understanding multidigit addition and subtraction as adding or subtracting like quantities (ones to ones, tens to tens, etc.) can facilitate the related later understandings that adding or subtracting decimal fractions or regular fractions must also involve adding or subtracting like quantities (for decimal fractions: tenths to tenths, hundredths to hundredths, etc. and for fractions: fourths to fourths, thirds to thirds, etc.). If whole number knowing and doing has
been a sense-making process intertwined with problem solving and explaining one’s thinking, it will be easier for students to make the necessary extensions and adjustments to their whole number knowledge as they enter these more advanced domains.

**Real-World Situations, Problem Solving, and Computation Need to be Continually Intertwined to Achieve Sense-Making and Computational Fluency**

The need to move to intertwining

Traditionally in the United States, computation of whole numbers has been taught first, and then problems using that kind of computation have been presented as applications. This approach has several problems. First, less-advanced students sometimes never reached the application phase, limiting greatly their learning. Second, word problems are usually put at the end of each computational chapter, so sensible students never read the problems: They just do the operation practiced in that chapter. This, plus the focus on teaching students to focus on key words in problems rather than building a complete mental model of the problem situation, leads to poor problem solving by students because they never learn to read and model the problems themselves. Third, seeing problem situations only after learning computations does not enable the meanings in the problem situations to become linked to those computations. This limits the meaningfulness of the computations and the ability of children to use the computations in a variety of situations.

Research has indicated that beginning with problem situations yields higher problem solving competence and equal or better computational competence. Children who start with problem situations directly model solutions to these problems. They later move on to more advanced mathematical approaches as they move through levels of solutions and of problem difficulty. Thus, the development of computational fluency and problem solving is intertwined when both are co-developing with understanding.

Research for many years has contrasted conceptual and procedural aspects of learning mathematics. Which should come first has been debated for a long time. The current state of the research presents a much more complex relationship between conceptual and procedural aspects than one preceding the other. Rather, they are continually intertwined and potentially facilitating each other. As a given child comes to understand more, the given method becomes more integrated within itself and in relation to other methods. As a method becomes more automatic, reflection about some aspect may become more possible, leading to a new understanding. These conceptual and procedural intertwinnings take place within individuals in individual ways. It may not even be useful to distinguish between these two aspects because doing and understanding are always intertwined in complex ways.

Furthermore, different researchers may refer to the same method as a procedure or as a concept, depending upon whether the focus is on carrying out the method or on the conceptual
underpinnings. And, in a given classroom at a given time, some students may be carrying out what looks like the same method, but they may well have different amounts of understanding of that method at that time. This is what the helping aspects of classroom teaching is all about--helping everyone to relate their methods to their knowledge in ways that gives them fluency and flexibility.

**The importance of children experiencing the range of real-world addition and subtraction situations to build meanings for addition and subtraction operations**

Researchers around the world have identified three main types of real-world addition and subtraction situations. Each type involves three different quantities (a total and two addends), and each of these quantities can be unknown. Learning to solve all of these problem variations by focusing on the problem meaning and modeling it, and eventually writing an algebraic equation with an unknown, provides vital experience in rich algebraic problem solving that can prepare all children for algebra. Research indicates that almost all of these problems are accessible to Grade 1 children and all can be mastered by Grade 2 children. However, typically in U.S. textbooks, only the simplest variation of each problem type has been included, those that most children can already solve in kindergarten. In contrast, in the texts of the Soviet Union, problems were given equally across the various types and unknowns, and 40% of first-grade problems and 60% of second-grade problems were 2-step problems (Stigler, Fuson, Ham, & Kim, 1986).

One type of addition and subtraction situation is what mathematicians call a binary operation situation: You have two groups of objects and you want to put them together (Put Together) or you have one group of objects and you want to take them apart (Break Apart) to make two groups. This type is sometimes called Combine; it may or may not have any physical action in it. For example, you may have 4 dogs and 3 cats and ask how many animals there are.

The second type is what mathematicians call a unary operation: You have one group of objects (an initial state) and you want to add a group of objects to it (Add To) or you want to take a group of objects away from it (Take Away). This type is sometimes called Change Plus or Change Minus because there is an initial state, a change (plus or minus), and a resulting state.

The third type compares two groups of objects to find out how much more (Compare More) or how much less (Compare Less) one group has than the other. There are many ways to ask the comparing sentence, and most ways have a pair of sentences that reverse the direction of comparison (A > B or B < A). This type is particularly difficult for young children, partly because how much more or less in not actually physically there in the situations. The language used is also frequently complex for children to learn, especially the direction of a given comparison.
There are also several different types of multiplication and division real-world situations. Some of these are accessible to children as young as kindergarten or even Pre-K if the language is simple, the numbers are small, and objects are available to model the problem.

Such situations in the form of word problems and real situations brought into the classroom by descriptions of students can provide contexts within which operations of addition, subtraction, multiplication, and division can come to take on their whole range of required mathematical meanings. These real-world meanings can be acted out, modeled with objects, and drawn with simplified math drawings. Students can tell and write as well as read and solve problems. Rich language use by retelling a story in your own words can build listening, vocabulary, and comprehension skills. Children in bilingual or ESL classes can learn mathematics in these ways.

There is an important distinction between a situation representation (an equation or a drawing) and a solution representation. The most powerful problem solving approach is to understand the situation deeply—draw it or otherwise represent it to oneself. This is the natural method used by young students. But textbooks, and teachers under their influence, push students to write solution representations that are not consistent with their view of the situation. Students will write $8 + \_ = 14$ for a problem like “Erica had $8$. She babysat last night and now has $14$. How much did she earn babysitting?” Textbooks often push students to write $14 - 8$, but this is not how most students will represent or solve that problem. Allowing students to represent the situation in their own way communicates that the goal of problem solving is to understand the problem deeply. With this view, students can experience success and move on to more difficult problems throughout their school and out-of-school life.

Learning progressions, multiple methods, and many algorithms

Over the past 30 years, there has been a huge amount of research all over the world concerning children’s solution methods in single-digit and multi-digit addition and subtraction. This research indicates that we need to change substantially our present classroom teaching and learning practices in these areas. For pre-K through Grade 2 children, most classrooms of children most of the time contain children who solve a given kind of problem in different ways, and most children have more than one available method for some kinds of problems. This is quite a different picture from the usual vision of mathematics problems as having one kind of solution that must be taught by the textbook and by the teacher and then learned and used by the children. This does not say that children learn these methods in an experiential vacuum. Their experiences are vital, and teachers and classroom experiences are central to enabling children to learn various methods.

Within each computational domain, individual learners move through progressions of methods from initial transparent, problem-modeling, concretely-represented methods to less
transparent, more problem-independent, mathematically sophisticated and symbolic methods. At a given moment, each learner knows and uses a range of methods that may vary by the numbers in the problem, by the problem situation, and by other individual and classroom variables. A learner may use different methods even on very similar problems, and any new method competes for a long time with older methods and may not be used consistently. Typical errors can be identified for each domain and for many methods. Ways to help students overcome these errors have been designed and studied. Detailed understandings of methods in each domain enable us to identify prerequisite competencies that can be developed in learners to make those methods accessible to all learners.

The constant cycles of mathematical doings and knowings in a given domain lead to learners’ construction of representational tools that are used mentally for finding solutions in that domain. For example, the counting word list initially is just a list of words used to find how many objects there are in a given group. Children use that list many times for counting, adding, and subtracting. Gradually, the list itself becomes a mathematical object. The words themselves become objects that are counted, added, and subtracted; other objects are not necessary. For students who have opportunities to learn with understanding, the written place-value notation can become a representational tool for multidigit calculations as the digits in various positions are decomposed or composed, and proportional statements can become a representational tool for solving a range of problems involving ratio and proportion.

Learners invent varying methods regardless of whether their classrooms have been focused on teaching for understanding or on rote memorizing of a particular method. However, a wider range of effective methods is developed in classrooms teaching for understanding. In classrooms in which teachers help students move through progressions to more advanced methods, children are more advanced. In rote classrooms, learner’s inventiveness is often focused on generating many different kinds of errors, most of which are partially correct methods created by a particular misunderstanding. Thus, even in traditional classrooms focused on memorizing standard computational methods, learners are not passive absorbers of knowledge. They build and use their own meanings and doings, and they generalize and reorganize these meanings and doings.

Because most people in the U.S. were taught only one method to do multidigit addition, subtraction, multiplication, and division, they think that there is only one way to do each of these (the way they were taught). These methods are called algorithms-- a general multistep procedure that will produce an answer for a given class of problems. Computers use many different algorithms to solve different kinds of problems. Inventing new algorithms for new kinds of problems is an increasingly important area of applied mathematics. Throughout history and at the present time around the world, many different algorithms have been invented and
taught for multidigit addition, subtraction, multiplication, and division. Different algorithms have been taught at different times in U.S. schools. Many immigrant children are taught one method at school and a different method at home, and children who immigrate are often forced to learn a new method that interferes with the old method they learned originally.

Each algorithm has advantages and disadvantages. Therefore, part of the decision-making around mastery for all concerns which algorithms might be supported in classrooms and the bases for selecting those algorithms. Research has now identified some accessible algorithms that are easier for children to understand and to carry out than the algorithms usually taught at present in the U.S. These are described more fully in Fuson (in press), Fuson and Burghardt (in press), and Fuson et al. (1997).

**Single-Digit Addition and Subtraction:**

**Much More Than “Learning the Facts”**

Learning single-digit addition and subtraction has for much of this century been characterized in the United States as “learning math facts.” The predominant learning theory was of these facts as rote paired-associate learning in which each pair of numbers was a stimulus (e.g., 7 + 6) and the answer (13) needed to be memorized as the response to this stimulus. “Memorizing the math facts” has been a central focus of the mathematics curriculum, and many pages of textbooks presented these stimuli to which children were to respond with their “memorized” response.

This view of how children learn basic single-digit computations was invalidated by one line of research earlier this century (by Brownell) and by much research from all over the world during the last thirty years. We now have very robust knowledge of how children in many countries actually learn single-digit addition and subtraction.

The unitary progression of methods used all over the world by children stems from the sequential nature of the list of number words. This list is first used as a counting tool, and then it becomes a representational tool in which the number words themselves are the objects that are counted (Bergeron & Herscovics, 1990; Fuson, 1986b; Steffe, Cobb, & Von Glasersfeld, 1998). Counting becomes abbreviated and rapid. Children can count on in addition situations (e.g., solve 8 + 6 by counting “8, 9, 10, 11, 12, 13, 14” rather than counting from 1 to 14). They also learn to count up to solve subtraction situations, which is much easier and much more accurate than is counting down, which is very subject to errors. So to solve 14 - 8, they think, “8 have been taken away, so 9, 10, 11, 12, 13, 14, that is 6 more left.” Children use this same strategy to solve unknown addend situations (8 + “ = 14). Some (or in some settings, many) children later go on to chunk numbers using thinking strategies, These chunkings turn additions children do not know into additions they do know (often using doubles). Of course, all
Achieving 21st Century Mastery for All

During this learning progression, children also learn some additions and subtractions automatically, especially for smaller numbers.

During this progression, which may last into third or even into fourth or fifth grade for some children (because they are not helped through the progression), individual children use a range of different methods on different problems. Learning disabled children and others having difficulty with math do not use methods that differ from this progression. They are just slower than others in this progression (Geary, 1994; Ginsburg & Allardice, 1984; Goldman, et al., 1988; Kerkman & Siegler, 1993).

Most of these methods are not ordinarily taught in the U.S. or in many other countries. However, when these more-advanced methods are not supported in the classroom, several years separates the earliest and latest users of advanced methods. In contrast, helping children progress through methods can lead all first graders to methods efficient enough to use for all of later multidigit calculation. Counting on can be made accessible to first graders; it makes rapid and accurate addition of all single-digit numbers possible. Single-digit subtraction is usually more difficult than is addition for U.S. children, primarily because children model taking away by inventing counting down methods, which are difficult and error-prone. Learning to think of subtraction as counting up to the known total, as is done in many other countries, makes subtraction as easy as, or easier than, addition. But at present, counting up to rarely appears in textbooks.

Children’s tools for beginning understandings of addition and subtraction are the counting word list (“one, two, three, four, etc.”), the ability to count objects, some indicating act (e.g., pointing, moving objects) tying words said and objects counted together (one at a time), and the count-cardinal knowledge that the last count word said tells how many objects there are in all. These tools are learned in the pre-school years by many but not all children in the United States. Focused help in pre-K and in kindergarten with all of this prerequisite knowledge could help all children come to mastery more rapidly. With these tools, addition can be done orally using concrete situations comprehensible to young learners. They count out objects for the first addend, count out objects for the second addend, and then count all of the objects (count all). This general counting all method then becomes abbreviated, interiorized, chunked, and abstracted, as discussed above.

The widely reported superiority of East Asian students over U.S. students in the early grades does not result from a focus on rote memorized addition and subtraction “facts.” It results from systematic visual and oral work that provides the underpinnings for strategies that are then explicitly taught. East Asian students, as well as students in many other parts of the world, are taught a general thinking strategy: making a ten by giving some from one addend
to the other addend. This method is facilitated by the number words in some countries (e.g., China, Japan, Korea, and Taiwan): “Ten, ten one, ten two, ten three, etc.”

Many of these children also learn numbers and addition using a ten-frame: an arrangement of small circles into 2 rows of 5. This pattern emphasizes 6, 7, 8, and 9 as 5 + 1, 5 + 2, 5 + 3, and 5 + 4. Work with this visual pattern enables many children to “see” these small additions under ten as using a 5-pattern. For example, some Japanese and Chinese adults report adding 6 + 3 by thinking/seeing 5 + 1 plus 3 = 5 + 4 = 9. This reduction of 6 + 3 to 5 + 1 + 3 = 5 + 4 = 9 is done very rapidly and without effort, as automatically as is recall. The ten-frame is also used to teach the “make a ten” method. For example, 8 has 2 missing in the ten-frame, so 8 + 6 requires 2 from the 6 to fill the ten-frame, leaving 4 to make 10 + 4 = 14. By the end of first grade, most children in these countries rapidly use these 5-patterns or ten-patterns to add single-digit numbers.

The “make a ten” method is also taught in some European countries. There are three prerequisites that children must learn in order to use the “make a ten” method effectively. They must know what number makes ten with each number (e.g., 10 = 8 + 2 or 6 + 4 or 7 + 3), be able to break apart any number into any of its two addends (to make ten and the rest over ten), and know 10 + n (e.g., 10 + 5 = 15). In countries that teach the “make a ten” method, these prerequisites are developed before the “make a ten” method is introduced. Many United States first and even second graders do not have these prerequisites consolidated, and they are rarely developed sufficiently in textbooks. Consequently few children invent this strategy. The strategy also is rarely taught in this country.

Textbooks in the United States typically show little understanding of children’s progression of methods. They move directly from counting all (e.g., 4 + 3 shows 4 objects and then 3 objects) to pages with only numbers, where children are to begin to “memorize their facts” (which they can not do because no answers are given). Children respond by inventing the experiential trajectory of methods discussed above, what I often call the “secret under-the-table world-wide progression of solution methods.”

This lack of fit between what is in textbooks and how children think is exacerbated by other features of textbook treatment of addition. Compared to other countries, the United States has had a very delayed placement of topics in the elementary school curriculum (Fuson, 1992a; Fuson, Stigler, & Bartsch, 1988). Almost all of first grade was spent on addition and subtraction below ten, and problems with totals above 12 were often in the last chapters (which many teachers never reached). Such simple problems were then also emphasized and reviewed in Grade 2, resulting overall in many more of the easier additions and relatively fewer of the more difficult single-digit additions (Hamann & Ashcraft, 1986). Thus, in contrast to East Asian children who are shown in first grade effective methods for solving the difficult additions
over ten (i.e., with totals between 10 and 18) in visually and conceptually supported ways, many U.S. children had little opportunity to solve such problems in first grade and were not supported in any effective methods to do so.

**Multidigit Addition and Subtraction: The Need for Using Accessible Algorithms**

There is considerable research on various ways in which children learn various multidigit addition and subtraction methods, though not nearly as much research as on single-digit addition and subtraction. In single-digit addition and subtraction, the same learning progression occurs in many different countries in spite of not being taught. Multidigit addition and subtraction depend much more on what is taught, and different children even within the same class may follow different learning progressions and use different methods. Multidigit addition and subtraction knowings seem to consist much more of different pieces that are put together in different orders and in different ways by different children (e.g., Hiebert & Wearne, 1986).

**Difficulties with words and numbers**

As with teen numbers, the English number words between 20 and 100 complicate the teaching/learning task for multidigit addition and subtraction. English names the hundreds and thousands regularly, but does not do so for the tens. For example, 3333 is said “3 thousand 3 hundred thirty 3" not “3 ten 3." English-speaking children must learn and then use a special sequence of decade words for 20, 30, 40, etc. This sequence, like the teens, has irregularities. Furthermore, teens words and decade words sound alike: In a classroom it is often difficult to hear the difference between “eighteen” and “eighty..” The same numbers 1 through 9 are re-used to write how many tens, hundreds, thousands, etc. Whether it is 3 tens or 3 hundreds or 3 thousands is shown by the relative position of the 3: how many positions to the left of the number farthest to the right is the 3? Relative position is a complex concept. French is even more complex, with its use of 20 as a base.

The written place value system is a very efficient system that lets us write very large numbers. But it is very abstract and misleading: The numbers in every position look the same. To understand the meaning of the numbers in the various positions, children need experience with some kind of size quantity supports (manipulatives) that show tens to be collections of ten ones and show hundreds to be simultaneously ten tens and one hundred ones, etc. Various kinds of such supports have been designed and used in teaching our written system of place value. However, classrooms rarely have enough of such supports for children themselves to use them (and many classrooms do not use anything). Such supports are rarely used in multidigit addition and subtraction, or they are used first alone to get answers without sufficient linking to a written method related to the manipulative method.
As a result, many studies indicate that U.S. children do not have or use quantity understandings of multidigit numbers (see reviews in Fuson, 1990, 1992a, 1992b). Instead, children view numbers as single digits side by side: 827 is functionally “eight two seven” and not 8 groups of 1 hundred, 2 groups of ten, and 7 single ones. Children make many different errors in adding and subtracting multidigit numbers, and many who do add or subtract correctly cannot explain how they got their answers.

Teaching for understanding and fluency

In contrast, research on instructional programs in the U.S., Europe, and South Africa indicate that focusing on understanding multidigit addition and subtraction methods results in much higher levels of correct multidigit methods and produces children who can explain how they got their answers using quantity language (Beishuizen, 1993; Beishuizen, Gravemeijer, & van Lieshout, 1997; Carpenter, Franke, Jacobs, & Fennema, 1998; McClain, Cobb, & Bowers, 1998; Fuson & Briars, 1990; Fuson & Burghardt, 1997, in press; Fuson, Smith, & LoCicero, 1997; Fuson, Wearne, et al., 1997). Characteristics of all of these approaches are that students used some kind of visual quantity support to learn meanings of hundreds, tens, and ones, and these meanings were related to the oral and written numerical methods developed in the classrooms.

Many different addition and subtraction methods were developed in these studies, often in the same classrooms (see Fuson, Wearne, et al., 1997, and Fuson & Burghardt, in press, for summaries of many methods). In most of these studies children invented various methods and described them to each other, but in some studies conceptual supports were used to give meaning to a chosen algorithm. Many studies were intensive studies of children’s thinking in one or a few classrooms, but some studies involved 10 or more classrooms including one study of all second-grade classrooms in a large urban school district (Fuson & Briars, 1990). In all studies a strong emphasis was placed on children understanding and explaining their method using quantity terms (e.g., using hundreds, tens, ones or the names of the object supports being used).

The function of quantity supports is to suggest meanings that can be attached to the written numbers and to the steps in the solution method with numbers. Therefore, methods of relating the quantity supports and the written number method through linked actions and through verbal descriptions of numerical method are crucial. However, in the classroom, supports often are used without recording anything except the answer at the end, and then students are shifted to written methods without linking to the steps taken with the supports. Thus, the written numerals do not necessarily take on meanings as tens, hundreds, etc., and the steps in the numeral method may be thought of as involving only single digits rather than their actual quantity meanings. This leaves them vulnerable to the many errors created by students without the meanings to direct or constrain them. Even for students who initially learn a meaningful
method, the appearance of multidigit numbers as single digits may cause errors to creep in. An important step in maintaining the meanings of the steps is to have students occasionally explain their method using the names for their quantity support (e.g., big cubes, etc., or money).

**Solution methods and accessible algorithms**

Many different methods of multidigit addition and subtraction are invented by children and are used in different countries. There is not space here to describe all of them or to analyze their respective advantages and disadvantages. However, 2 addition methods and 1 subtraction method have been selected for discussion. These are especially clear conceptually, are easy for even less-advanced students to carry out, and are less prone to errors than are many other methods. We also show the addition and the subtraction algorithms that are currently taught most frequently in textbooks in the U.S.

In Figure 3 the algorithm on the top left is the addition method currently appearing in most U.S. textbooks. It starts at the right, in contrast to reading, which starts at the left. Most methods that children invent start at the left, perhaps because of reading from the left and perhaps because we read our number words starting at the left. The current addition algorithm has 2 major problems. One is that many children object initially (if they are in a position in which mathematical objections can be voiced) to putting the little 1s above the top number. They say that you are changing the problem. And in fact, this algorithm does change the numbers it is adding as it goes by adding in these carries to the top. The second method in the top row of Figure 3 does not change the top number: the new 1 ten is written down in the space for the total on this line (children using base-ten blocks in Fuson & Burghardt, 1993, in press, invented this method so that they did not change the answer as they went). It is also easier to see the total 14 ones when the 1 is written so close to the 4. The second problem with the present U.S. algorithm is that it makes single-digit addition very difficult. You must add in the 1 to the top number, remember it even though it is not written, and add that remembered number to the bottom number. If instead you add the two numbers you see, you may forget to go up to add on the 1 ten (or 1 hundred). The second method solves this problem: You just add the two numbers you see and then increase that total by 1. This makes the adding much easier for less-advanced children.

Method B in Figure 3 separates the two major steps in multidigit adding. The total for adding each kind of multiunit is written on a new line, emphasizing that you are adding each kind of multiunit. The carrying-regrouping-trading is just done as part of the adding of each kind of multiunit: the new 1 ten of the next larger multiunit is just written in the next-left column. One then does the final step of multidigit adding: add all of the partial additions to find the total. Method B can be done in either direction (Figure 3 shows the left-to-right version). Because
you write out the whole value of each addition (e.g., 500 + 800 = 1300), this method facilitates children’s thinking about and explaining how and what they are adding.

The drawings at the far right can be used with any of the 3 methods shown to support understanding of the major components of the methods. The different sizes of the ones, tens, and hundreds in the drawings support children’s adding of those like quantities to each other. Ten of a given unit can be encircled to make 1 of the next higher unit (10 ones = 1 ten, 10 tens = 1 hundred, 10 hundreds = 1 thousand). The issue for each algorithm then is how to record the adding of each kind of unit, the making of each 1 new larger unit from 10 of the smaller units, and adding of the partial additions to make the total. Circling the new ten units can also support the general “make a ten” single-digit methods.

Under the drawing are summarized the two vital elements of using drawings or objects to support understanding of addition methods. First is a long stage 1 in which the objects or drawings are linked to the steps in the algorithms to give meanings to the numerical notations in those algorithms. A second but crucial stage 2 then lasts over an even longer period (over years) in which students only carry out the numerical algorithm but they occasionally explain it using words describing quantity objects or drawings so that meanings stay attached to the steps of the algorithm. Stage 2 is vital because of the single-digit appearance of our written numerals; these do not direct correct methods or inhibit incorrect methods, as the objects and drawings do, and errors can creep into understood methods, especially as children learn other solution methods in other domains.

Two subtraction methods are shown in Figure 3. The method on the left is the most widely used current U.S. algorithm. It moves from right to left, and it alternates between the two major subtraction steps: Step 1: regrouping (borrowing, trading) to get 10 more of a given unit so that unit can be subtracted (necessary when the top unit is less than the bottom unit) and Step 2: subtracting after the top number has been fixed. The regrouping may be written in different ways (e.g., as a little 1 beside the 4 instead of crossing out the 4 and writing 14 above). The alternating between the two major subtracting steps presents three kinds of difficulties to students. One is initially learning this alternation. Two is then remembering to alternate the steps. The third is that the alternation renders students susceptible to the pervasive subtracting error: subtracting a smaller top number from a larger bottom number (e.g., doing 62 - 15 as 53). When moving left using the current method, a solver sees two numbers in a column while primed to subtract. For example, after regrouping in 1444 - 568 to get 14 in the rightmost column and subtracting 14 - 6 to get 8, one sees 3 at the top and 6 at the bottom of the next column. Automatically the answer 3 is produced (6 - 3 = 3). This answer must be inhibited while one thinks about the direction of subtracting and asks whether the top number is larger than the bottom (i.e., asks oneself whether regrouping or borrowing is necessary).
The accessible subtraction method shown in the bottom middle of Figure 3 separates the two steps used in the current method. First, a student asks the regrouping (borrowing) question for every column, in any direction. The goal here is to rewrite the whole top number so that every top digit is larger than the bottom digit. This makes the conceptual goal clearer: You are rearranging the units in the top number so that they are available for subtracting like units. It also prevents the ubiquitous top-from-bottom error because you fix everything before doing any subtracting. Doing the fixing in any direction allows children to think in their own way. The second major step then is to subtract every column. This also can be done in any order.

The drawing at the bottom right of Figure 3 shows how a size drawing or size objects can support the two aspects of multidigit subtracting. There are not enough ones, or tens, or hundreds to do the needed subtracting, so 1 larger unit is opened up to make 10 of the needed units. The subtraction can be done from this 10, facilitating the “take from ten” single-digit subtraction method. Or students can count up to find the difference in the written number problem.

The irregular structure of the English words between twenty and ninety-nine continues to present problems in multidigit problems because all single-digit and multidigit calculation is done using the words as oral intermediaries for the written numbers, and these words do not show the tens in the numbers. Using English forms of the regular East Asian words (“1 ten 4 ones” for 14) along with the ordinary English number words has been reported to be helpful (Fuson, Smith, & Lo Cicero, 1997). This permits children to generalize single-digit methods meaningfully. For example, for 48 + 36, students can use their single-digit knowings and think, “4 tens + 3 tens is 7 tens” rather than having to think “forty plus thirty is ?” or use only single-digit language (“four plus three is seven”), and thus ignoring the values of the numbers.

**Textbook and curricular issues**

U.S. textbooks have several problematic features that complicate children’s learning of multidigit addition and subtraction methods. The grade placement of topics is delayed compared to that of other countries (Fuson, Stigler, & Bartsch, 1988), and problems have one more digit each year so that this topic continues into Grades 5 or even 6. In contrast, multidigit addition and subtraction for large numbers are completed in some countries by Grade 3. In the first grade in the U.S., 2-digit addition and subtraction problems with no regrouping (carrying or borrowing) are given but no problems requiring regrouping are given until almost a year later, in second grade. Problems with no regrouping set children up for making the most common errors, especially subtracting the smaller number from the larger even when the larger number is on the bottom (e.g., 72 - 38 = 46). This error is one major reason that on standardized tests only 38% of U.S. second graders are accurate on problems such as 72 - 38.
Accessibility studies indicate that first graders can solve 2-digit addition problems with trading if they can use drawings or quantity supports (Fuson, Smith, & Lo Cicero, 1997; Carpenter, Franke, Jacobs, & Fennema, 1998). Because knowing when to make 1 new ten is an excellent use of place-value knowledge, such problems can be thought of as consolidating place-value ideas, not just as doing addition. Giving from the beginning subtraction problems that require regrouping would help children understand the general nature of 2-digit subtraction. This might well be delayed until second grade because children find 2-digit subtraction much more difficult than addition. But second graders learning with quantity supports and with a focus on understanding their methods can have high levels of success.

This review suggests some central features for effective reform and traditional texts. Any algorithms that are included need to be accessible to children and to teachers, and support needs to be provided so that they are learned with understanding. The research-based accessible methods in Figure 3 were included here to indicate algorithms that are more accessible than those presently appearing in most U.S. textbooks. Further, children need to use quantity supports in initial experiences with multidigit solving and multidigit algorithms so that these can be learned with meaning. Finally, students and teachers need to use referents when discussing methods so that everyone can follow the discussion. Drawing quantities can be helpful here.

**Conclusion**

Recent research clearly indicates that non-traditional approaches can help U.S. children come to carry out, understand, and explain methods of multidigit addition and subtraction rather than only carry out a method. This higher level of performance can also be done at earlier grades than is presently expected only for answers. Features of classrooms engendering this higher level of performance are: an emphasis on understanding and explaining methods; initial use by children of quantity supports or drawings that show the different sizes of ones, tens, and hundreds in order to give meanings to methods with numbers; and sufficient time and support for children to develop meanings for methods with numbers and for prerequisite understandings (these may be developed alongside the development of methods) and to negotiate and become more skilled with the complexities of multistep multidigit methods.

The most effective approach at present seems to be to make the learning of algorithms more mathematical by considering it an important arena of mathematical pattern-finding and invention that will use and contribute to robust understandings of our place value system of written numeration. Meaningful discussion of various standard algorithms brought into the classroom from children’s homes (e.g., the subtraction algorithm widely used in Latin America and Europe, see Ron, 1998) has an important role. Seeking to discover why each one works provides excellent mathematical investigations. It also seems to be important to share accessible
methods with less-advanced children so that they have a method they understand and can use. However, the focus should be on their understanding and explaining, not just on rote use.

All three of the accessible methods in Figure 3 were invented by children but also have been shared with and learned meaningfully by many children. There may well be other methods not as yet discovered (or rediscovered) that will be even more powerful. Comparing methods to see how they take care of the crucial issues of that domain facilitates reflection by everyone on the underlying conceptual and notational issues of that domain. This seems a much more appropriate focus in the 21st century, where new machine algorithms will be needed and new technology will require many people to learn complex multistep algorithmic processes. If this focus is accompanied by a continual focus on testing and teaching accessible methods as well as on fostering invention, all children should be able to learn and explain a multidigit addition and subtraction method as well as carry it out accurately.

Design Principles and a Proposal for Mastery Curricula and Mastery Teaching

One of the major themes in this chapter has been the need to reorganize U.S. curricula into coherent chunks and spend more time on these chunks. A specific research-based proposal about how to do this will now be briefly outlined. This proposal is not the only way that coherent foci could be developed. But it is an obvious one based on present research. It has also been worked out in considerable detail in my own classroom curricular research, and developing versions of it have been used in a wide range of classrooms. This proposal is overviewed in Tables 1 and 2 and in Figure 3. Table 1 outlines design principles concerning aspects of the coherent chunks and how they are related. Table 2 further specifies goals at each grade level from Pre-K through Grade 2 to indicate how these goals build from grade to grade without excessive repetition.

Figure 3 expresses in a schematized fashion the seven coherent chunks. The foundational chunk is numeration, which relates quantities in the world, number words, and number notation. Algebraic problem solving forms the core of the program. It relates the three major types of real-world addition/subtraction situations (in ellipses in Figure 3). Multiplication/division (in rectangles) form surrounding areas. The four types pictured here indicate both how multiplication/division grows out of and is related to addition/subtraction and how complex multiplication/division is (it has four different types of situations). Though some elementary ideas in these multiplication/division domains can be introduced early, it is much more cohesive and developmentally appropriate to concentrate heavily and deeply on the whole range of addition/subtraction situations and obtain mastery on them in Pre-K to Grade 2. In Grade 3 and beyond students can then begin substantive and deep examination of the whole range of multiplication/division situations and the extensions to rational numbers and integers. Likewise, some substantive works in geometry, measure, and data are sensible for Pre-K
through Grade 2. But again only some small foundational work will be done in the early grades. Serious deeper mathematical work in these domains is appropriate beginning in Grade 3.

The starred items in Table 2 underscore a chief impediment to mastery learning by all: our present inability to reach all children in the prekindergarten years. Students who are not reached in prekindergarten may enter kindergarten with much less relevant mathematical experiences and therefore may not be ready for kindergarten learning at the beginning of the year. Similarly, those who do not have kindergarten may not be ready for Grade 1 work. This places very considerable extra burdens on kindergartens and Grade 1 classrooms. It thus is crucial for Headstart and other prekindergarten programs to be organized to meet the prekindergarten goals. Efforts must also be made to reach parents who do not send their children to preschools so that they can carry out mathematical activities at home. Even with these efforts, underserved children may need extra time to catch up and learn the ambitious goals. Some ways of ensuring that they do have such extra time to reach ambitious goals need to be found rather than sending them on to Grades 1 or 2 underprepared.

It should be underscored that the concentration of time and energy on these seven chunks enables considerably more to be learned in all areas by the end of Grade 2 than is the case at present. Thus the foundation will be stronger and will also enable more substantive mathematics to be learned in Grades 3 and above.

**Conclusion**

A new more complex view of goals and standards for Pre-K through Grade 2 is necessary to achieve the new more complex goals of mathematics learning and teaching necessary for the 21st century. These goals and standards require focused concentrated ambitious grade level topics that build across the age levels. Replacing “standardized” testing with mastery testing linked to coherent and ambitious grade-level goals can help focus everyone’s energies on the same developmentally appropriate teaching/learning goals and facilitate mastery for all. Some children may need additional learning time and support to achieve mastery goals. This is more equitable than the present practice of allowing children to move on with quite different amounts of learning and falling increasingly behind. Instruction organized into an orienting, supporting, learning, and long-term remembering phases can facilitate learning by all. Real-world situations, problem solving, and computation need to be continually intertwined to achieve sense-making and computational fluency. Children can be helped to learn more advanced and rapid single-digit methods that can also be understood. Accessible addition and subtraction algorithms can enable more children to learn with understanding and achieve high levels of mastery. Finally, a specific proposal to organize the Pre-K through Grade 2 curriculum into seven research-based coherent chunks was outlined. This permits deep coherent learning in each of the areas with strong articulation and building across the grade levels. All of these
attributes summarized here can coalesce into truly achieving mastery by all children of ambitious world-class goals by the end of Grade 2.
References


Achieving 21st Century Mastery for All


Real-world situations, problem solving, and computation need to be continually intertwined in order to achieve sense-making and computational fluency.

Each of the three types of addition/subtraction real-world situations (Break Apart/Put Together, Get More/Give Away, Compare) forms one of the coherent curricular chunks (chunks 2, 3, and 4). Each of these is also integrated with other related topics.

Algebraic problem solving (chunk 5) relates all of these by coordinating children’s
a) use of situation drawings,
b) reading/writing/solving of word problems of each of the three types using varying language,
c) use of equations with one unknown that varies (by grade 2) across all of the 3 possible unknowns,
d) explaining solution methods.

Computation is integrated with problem solving throughout chunks 2, 3, 4, and 5, but is also a special focus of numeration activities in chunk 1.