Geometric and Spatial Thinking in Early Childhood Education

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Geometry and spatial reasoning are inherently important because they involve “grasping...that space in which the child lives, breathes and moves...that space that the child must learn to know, explore, conquer, in order to live, breathe and move better in it” (Freudenthal, in National Council of Teachers of Mathematics, 1989, p. 48). In addition, especially for early childhood, geometry and spatial reasoning form the foundation of much learning of mathematics and other subjects.

Although our knowledge of young children’s geometric and spatial thinking is not as extensive as their numerical thinking, it has grown substantially and can be used as a basis for curriculum development and teaching. Here, we briefly review these two main areas of this research: shape and transformation (2-D figures; angle; 3-D figures; congruence, symmetry, and transformations; composition and decomposition) and spatial thinking (spatial orientation: maps and navigation; and spatial visualization and imagery). We conclude with implications for curriculum and instruction in early childhood geometry.

Shape and Transformation

2-D Geometric Figures

Too often, teachers and curriculum writers assume that children in early childhood classrooms have little or no knowledge of geometric figures. One early study found that kindergarten children had a great deal of knowledge about shapes and matching shapes before instruction began. Teachers tended to elicit and verify this prior knowledge but did not add content or develop new knowledge. That is, about two-thirds of the interactions had children repeat what they already knew in a repetitious format as in the following exchange (Thomas, 1982).

Teacher: Could you tell us what type of shape that is?
Children: A square.
Teacher: Okay. It’s a square.

Teachers’ additions were frequently incorrect (all rhombi are squares) or unfortunate (child gets a square for a rectangle, teachers asks for a rectangle again), or that two triangles put together always make a square or a square cut in half always yields two triangles. A high proportion of questions were closed and the majority of responses were at the memory level. Few children asked questions, but when they did, the teacher responded with silence (Thomas, 1982).

A more recent study confirmed that current practices in the primary grades also promote little conceptual change: First grade students in one study were more likely than older children to differentiate one polygon from another by counting sides or
vertices (Lehrer, Jenkins, & Osana, 1998). Over time, children were less likely to notice these attributes, given conventional instruction of geometry in the elementary grades.

Such neglect evinces itself in student achievement. Students are not prepared for learning more sophisticated geometry, especially when compared to students of other nations (T. P. Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1980; Fey et al., 1984; Kouba et al., 1988; Stevenson, Lee, & Stigler, 1986; Stigler, Lee, & Stevenson, 1990). In the recent TIMMS work, U.S. students scored at or near bottom in every geometry task (Beaton et al., 1996; Lappan, 1999).

Such comparisons may be present even among preschoolers in various countries (Starkey et al., 1999). On a geometry assessment, 4-year-olds from America scored 55% compared to those from China at 84%. Thus, cultural supports are lacking from the earliest years in the U.S.

How do children think and learn about shapes? It is possible they are born with a tendency to form certain mental prototypes. People in a Stone Age culture with no geometric concepts were asked to choose a “best example” of a group of shapes, such as a group of quadrilaterals and near-quadrilaterals (Rosch, 1975). People chose a square and circle more often, even when close variants were in the group. For example, the group with squares included square-like shapes that were not closed, had curved sides, and had non-right angles. So, we might have “built-in” preferences for closed, symmetric shapes (c.f. Bornstein, Ferdinandsen, & Gross, 1981).

Culture shapes these preferences. We conducted an extensive examination of materials that teach children about shapes from books, toy stores, teacher supply stores, and catalogs. With few exceptions (and with signs that this is changing in recent years), these materials introduce children to triangles, rectangles, and squares in rigid ways. Triangles are usually equilateral or isosceles and have horizontal bases. Most rectangles are horizontal, elongated shapes about twice as long as they are wide. No wonder so many children, even throughout elementary school, say that a square turned is “not a square anymore, it’s a diamond” (c.f. Lehrer et al., 1998). Research indicates that such rigid visual prototypes can rule children’s thinking throughout their lives (Burger & Shaughnessy, 1986; Fisher, 1978; Fuys, Geddes, & Tischler, 1988; Kabanova-Meller, 1970; Vinner & Hershkowitz, 1980; Zykova, 1969).

Specifically, what visual prototypes and ideas do preschool children form about common shapes? Decades ago, Fuson and Murray (1978) reported that by 3 years of age over 60% of children could name a circle, square, and triangle. More recently, Klein, Starkey, and Wakeley (Klein, Starkey, & Wakeley, 1999) reported shape naming accuracy of 5-year-olds as: circle, 85%, square, 78%, triangle, 80%, rectangle, 44%.

We recently conducted several studies with hundreds of children, ages 3 to 6 years. In the first study (Clements, Swaminathan, Hannibal, & Sarama, 1999), we used the

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1 Of course, all physical shapes are 3-D; however, we will follow common usage in referring to, for instance, an illustration of a 2-D shape or a triangle pattern block (instead of a pattern block with a triangle face).
same line drawings we previously used with elementary students for comparison purposes. Children identified circles quite accurately: 92%, 96%, and 99% for 4-, 5-, and 6-year-olds, respectively. Only a few of the youngest children chose the ellipse and curved shape (Fig. 1). Most children described circles as "round," if they described them at all. Thus, the circle was easily recognized but relatively difficult to describe for these children. Evidence suggests that they matched the shapes to a visual prototype.

Figure 1. Student marks circles. Adapted from (Razel & Eylon, 1991)

Children also identified squares fairly well: 82%, 86%, and 91% for 4-, 5-, and 6-year-olds, respectively. Younger children tended to mistakenly choose nonsquare rhombi ("diamonds" such as No. 3 in Fig. 2). However, they were no less accurate in classifying squares without horizontal sides (No. 5 and 11). Children were more likely to be accurate in their square identification when their justifications for selection were based on the shape's attributes (e.g., number and length of sides).
They were less accurate recognizing triangles and rectangles. However, their scores were not low; about 60% correct for triangles (see Fig. 3). Children's visual prototype seems to be of an isosceles triangle.

Figure 3. Student marks triangles. Adapted from (Burger & Shaughnessy, 1986) and (Clements & Battista, 1991)
Young children tended to accept “long” parallelograms or right trapezoids (shapes 3, 6, 10, and 14 in Fig. 4) as rectangles. Thus, children's visual prototype of a rectangle seems to be a four-sided figure with two long parallel sides and “close to” square corners.

*Figure 4.* Student marks rectangles. Adapted from (Burger & Shaughnessy, 1986) and (Clements & Battista, 1991)
Although young children in this study were less accurate recognizing triangles and rectangles, they are not remarkably smaller than those of elementary students (Clements & Battista, in press) as shown in Fig. 5 and 6 (in addition, many of the elementary students were from relatively high SES populations). Indeed, for all shapes assessed two trends were evident. First, as discussed previously, very young children possess knowledge of geometric figures. Second, children show a steady, but hardly remarkable, improvement from preschool through the elementary grades.
In the second study (Hannibal & Clements, 2000), we asked children ages 3 to 6 to sort a variety of manipulative forms. We found that certain mathematically irrelevant characteristics affected children’s categorizations: skewness, aspect ratio, and, for
certain situations, orientation. With these manipulatives, orientation had the least effect. Most children accepted triangles even if their base was not horizontal, although a few protested. Skewness, or lack of symmetry, was more important. Many rejected triangles because “the point on top is not in the middle.” For rectangles, on the other hand, many children accepted non-right parallelograms and right trapezoids. Also important was aspect ratio, the ratio of height to base. Children preferred an aspect ratio near one for triangles; that is, about the same height as width. Other forms were “too pointy” or “too flat.” Children rejected both triangles and rectangles that were “too skinny” or “not wide enough.”

These are simple tasks, chosen initially for their consistency with other research-based tasks and traditional curricular goals. Yet they do illustrate both the strength of children’s initial competencies and the weakness of the cultural and instructional support for building upon them. Further, children’s capabilities exceed naming, describing, and sorting shapes. We turn to additional aspects of children’s knowledge of shape and spatial structure.

Angle and Turn

Angles are turning points in the study of geometry and spatial relationships. Unfortunately, one does not have to turn far for examples of children’s difficulty with the angle concept (Lindquist & Kouba, 1989). Children have many different ideas about what an angle is. These ideas include “a shape,” a side of a figure, a tilted line, an orientation or heading, a corner, a turn, and a union of two lines (Clements & Battista, 1990). Students do not find angles to be salient attributes of figures (Clements, Battista, Sarama, & Swaminathan, 1996; M. C. Mitchelmore, 1989). When copying figures, students do not always attend to the angles.

Similarly, regarding the size of angles, children frequently focus on the length of the line segments that form its sides, the tilt of the top line segment, the area enclosed by the triangular region defined by the drawn sides, the length between the sides, or the proximity of the two sides (Clements & Battista, 1989). Some misconceptions decrease over the elementary years, such as orientation; but others, such as the effect of segment length, do not change, and some, such the distance between end points, increase (Lehrer et al., 1998).

Nevertheless, there are some initial competencies on which instruction might build. Preschoolers use angles intuitively in their play, such as block building (Ginsburg et al., 1999). In an early study, while 5-year-olds showed no evidence of attention to angle in judging congruence, they could match angles in correspondence tasks (Beilin, 1984; Beilin, Klein, & Whitehurst, 1982). Some primary-grade children can distinguish between angles based on size (Lehrer et al., 1998).

There is some research on instructional approaches that attempt to develop these early abilities. One uses multiple concrete analogies (M. C. Mitchelmore, 1993). Practical experience in various situations (e.g., turns, slopes, meetings, bends, directions, corners, opening) helps children understand angular relationships in each situation individually. Gradually, children develop general angle concepts by recognizing
common features of these situations. Research on teaching activities based on these ideas revealed that most elementary age students understood physical relations. Turn, or rotation, was a difficult concept to understand in concrete physical contexts.

Other research supports the importance of integration across situations and ideas. One study took as the starting point children’s experience with physical rotations, especially rotations of their own bodies (Clements et al., 1996). During the same time, they gained limited knowledge of assigning numbers to certain turns, initially by establishing benchmarks. A synthesis of these two domains—turn-as-body-motion and turn-as-number—constituted a critical juncture in learning about turns for many elementary students. This and other studies have used the Logo turtle to help children mathematize2 their physical experiences.

Related topics include parallel and perpendicular lines. Both are difficult concepts for students in some applications. However, children as young as 3 (Abravanel, 1977) and 4 years use parallelism in alignment tasks and 6-year-olds can name parallel and non-parallel lines, although they have difficulty locating parallels in complex figures (M. Mitchelmore, 1992).

Teaching Australian grade 1 students about perpendicular lines was abandoned because students were unable to conceptualize perpendiculars as lines in a special angular relationship (M. Mitchelmore, 1992). However, as noted, preschoolers deliberately use parallelisms and perpendicularly intuitively in their block building play (Ginsburg et al., 1999). It remains to be seen if curricula and teaching approaches that build on these early beginnings can effectively facilitate lasting learning outcomes.

3-D Figures

Similar to findings regarding two-dimensional figures, students do not perform well with three-dimensional shapes. Most intermediate-grade students have difficulty naming solids (Thomas P. Carpenter, Coburn, Reys, & Wilson, 1976). South African first graders used different names for solids (such as “square” for cube) but were capable of understanding and remembering features they discussed (Nieuwoudt & van Niekerk, 1997). U.S. students’ reasoning about solids was much like that about plane figures; they referred to a variety of characteristics, such as “pointyness” and comparative size or slenderness (Lehrer et al., 1998). Students also treated the solid wooden figures as malleable, suggesting that the rectangular prism could be transformed into a cube by “sitting on it.”

Use of plane figure names for solids may indicate a lack of discrimination between two and three dimensions (Thomas P. Carpenter et al., 1976). Learning only plane figures in textbooks during the early primary grades may cause some initial

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2 We define mathematization as representing and elaborating mathematically— creating models of an everyday activity with mathematical objects, such as numbers and shapes; mathematical actions, such as counting or transforming shapes; and their structural relationships. Mathematizing involves reinventing, redescribing, reorganizing, quantifying, structuring, abstracting, and generalizing that which is first understood on an intuitive and informal level in the context of everyday activity.
difficulty in learning solids. Construction activities involving nets (foldout shapes of solids) may be valuable as they require children to switch between more-analytic 2-D and synthetic 3-D situations (Nieuwoudt & van Niekerk, 1997).

Congruence, Symmetry, and Transformations

Young children develop beginning ideas not just about shapes, but about congruence and transformations. Although many young children judge congruence (Are these two shapes “the same”?) based on whether they are, on the whole, more similar than different (Vurpillot, 1976), even 4-year-olds and some younger children can generate strategies for verifying congruence for some tasks. Preschoolers often try to judge congruence using an edge matching strategy, although only about 50% can do it successfully (Beilin, 1984; Beilin et al., 1982). They gradually develop a greater awareness of the type of differences between figures that are considered relevant and move from considering various parts of shapes to considering the spatial relationships of these parts (Vurpillot, 1976). In about first grade, they consider both multiple attributes and their spatial relationships and begin to use superposition. Thus, strategies supercede one another in development (e.g., motion-based superposition) becoming more powerful, sophisticated, geometrical, and accurate.

Other studies have focused on geometric motions. Some have reported that younger students’ abilities are slight. For example, one study showed that second graders learned manual procedures for producing transformation images but did not learn to mentally perform such transformations (Williford, 1972). In contrast, other studies indicate that even young children can learn something about these motions and appear to internalize them, as indicated by increases on spatial ability tests (Clements, Battista, Sarama, & Swaminathan, 1997; Del Grande, 1986). Slides appear to be the easiest motions for students, then flips and turns (Perham, 1978); however, the direction of transformation may affect the relative difficulty of turn and flip (Schultz & Austin, 1983). Results depend on specific tasks, of course; even 4- to 5-year olds can do turns if they have simple tasks and orientation cues (Rosser, Ensing, Glider, & Lane, 1984). Further, some studies indicate that second-grade students are capable of mental rotation involving imagery (Perham, 1978; Rosser, Lane, & Mazzeo, 1988).

Under the right conditions, children of all ages can apply similarity transformations to shapes. Even 4 and 5 year-olds can identify similar shapes in some circumstances (Sophian & Crosby, 1998). The coordination of height and width information to perceive the proportional shape of a rectangle (fat vs. skinny, wide or tall) might be a basic way of accessing proportionality information. This may serve as a foundation for other types of proportionality, especially fractions. Similarly, other research shows first graders can engage in and benefit from similarity tasks (Confrey, 1992).

Children have intuitive notions of symmetry from the earliest years (Vurpillot, 1976). Symmetric stimuli are not only preferred but are consistently detected faster, discriminated more accurately, and often remembered better than asymmetrical ones. Preference for vertical symmetry develops between 4 and 12 months of age (Bornstein et al., 1981) and vertical bilateral symmetry remains easier for students to handle than
horizontal symmetry (Genkins, 1975). However, many concepts of symmetry are not firmly established before 12 years of age (Genkins, 1975). Julie Sarama has noticed that children often use and refer to rotational symmetry as much as they do line symmetry in working with pattern blocks (Sarama, Clements, & Vukelic, 1996).

Computer environments can be particularly helpful in learning congruence, transformations, and symmetry (Clements & Battista, in press). Indeed, the effects of Logo microworlds on symmetry were particularly strong for young (kindergarten) students. Writing Logo commands for the creation of symmetric figures, testing symmetry by flipping figures via commands, and discussing these actions apparently encouraged students to build richer and more general images of symmetric relations (with possibly some overgeneralization). Students had to abstract and externally represent their actions in a more explicit and precise fashion in Logo activities than, say, in free-hand drawing of symmetric figures.

Composition and Decomposition

Another of the many processes young children can perform with geometric shapes is composition. I will take this opportunity to both overview the research on children's composing and decomposing competencies and illustrate how research can be used to go beyond “checklist” approaches to curriculum. The following is a research-based developmental sequence that approximately spans ages 4 to 8 years. The basic competence is combining shapes to produce composite shapes³. At each level, a child does the following.

1. Manipulates shapes as individuals, but is unable to combine them to compose a larger shape.

1. Similar to step 1, but can concatenate shapes to form a picture. Each shape represents a unique role, or function in the picture. Can fill simple frames using trial and error (Mansfield & Scott, 1990; Sales, 1994). Uses turns or flips to do so, but again by trial and error; cannot use motions to see shapes from different perspectives (Sarama et al., 1996). Thus, children at steps 1 and 2 see no geometric relationship between shapes or between parts of shapes (i.e., a property of the shape), at least in the context of these activities. Many 4-year-olds are at this level.

2. Matches shapes using gestalt configuration or one component such as side length (Sarama et al., 1996). If several sides of the existing arrangement form a partial boundary of a shape, the child can find and place that shape. If such cues are not present, the child matches by a side length. The child may attempt to match corners, but does not possess angle as a quantitative entity, so will try to match shapes into corners of existing arrangements in which their angles do not fit. Rotation and flipping are used, usually by trial-and-error, to try different arrangements (a “picking and discarding” strategy). Thus, there is intentionality and anticipation (“I know what will fit”), based on shapes’ components.

³ The notion of creating and then iterating units and higher-order units to construct patterns, measure, or compute has been established as a basis for mathematical understanding and analysis (Steffe & Cobb, 1988)
3. Matches shapes using angles as well as side lengths. Eventually considers several alternative shapes with angles equal to the existing arrangement. Rotation and flipping are used intentionally (and mentally, i.e., with anticipation) to select and place shapes (Sarama et al., 1996). Can fill complex frames (Sales, 1994) or cover regions (Mansfield & Scott, 1990). Beginning to form substitution relationships among shapes (e.g., two pattern block trapezoids make a hexagon).

4. Forms composite units of shapes by trial-and-error (Clements, Battista, Sarama, & Swaminathan, 1997). May combine these composite units by simple duplication.

5. Constructs and operates on composite units intentionally (i.e., students conceptualize each unit as being constituted of multiple singletons and as being one higher-order unit). Can continue a pattern of shapes that leads to a “good covering,” but without coordinating units of units.

6. Builds and applies units of units (superordinate units). For example, in constructing spatial patterns, students extend their patterning activity to create a tiling with a new unit shape—a (higher-order) unit of unit shapes that they recognize and consciously construct.

These levels represent a synthesis across divergent studies. We are in the process of empirically evaluating the validity of this sequence with a cross-sectional approach and designing off- and on-computer activities for each level for evaluation with teaching experiments.

Implications for Theory

According to the theory of Pierre and Dina van Hiele, students progress through qualitatively distinct levels of thought in geometry (van Hiele, 1986; van Hiele-Geldof, 1984). At Level 1, called the “visual” level, students can only recognize shapes as wholes and cannot form mental images of them. A given figure is a rectangle, for example, because “it looks like a door.” They do not think about the defining attributes, or properties, of shapes. At level 2, descriptive/analytic, level students recognize and characterize shapes by their properties. For instance, a student might think of a square as a figure that has four equal sides and four right angles. Properties are established experimentally by observing, measuring, drawing, and model-making. Students find that some combinations of properties signal a class of figures and some do not; thus the seeds of geometric implication are planted. Students at this level do not, however, see relationships between classes of figures (e.g., a student might believe that a figure is not a rectangle because it is a square). Many students do not reach this level until middle or even high school.

Research reviewed here suggests that it is not viable to conceptualize a purely visual level (1), followed and replaced by a basically verbal descriptive level (2) of geometric thinking—a common interpretation. Instead, different types of

We reserve the term “property” for those attributes that indicate a relationship between parts, or components, of shapes. Thus, parallel sides, or equal sides, are properties. We use “attributes” and features interchangeably to indicate any characteristic of a shape, including properties, other defining characteristics (e.g., straight sides) and nondefining characteristics (e.g., “right-side up”).
reasoning—those characterizing different levels—can coexist in an individual and can be developing simultaneously but at different rates. From this perspective, levels do not consist of unadulterated knowledge of only one type. This view is consistent with recent literature from Piagetian and cognitive traditions (e.g., Minsky, 1986; Siegler, 1996; Snyder & Feldman, 1984), as well as re-interpretations of van Hielian theory, including Clements (1992), which we are elaborating here, and others (Gutiérrez, Jaime, & Fortuny, 1991; Lehrer et al., 1998; Pegg & Davey, 1998). These interpretations reject the assumption that one level of geometric knowledge, such as visual knowledge, exists in an individual. Instead, for example, children at both levels 1 and 2 possess triangle schemes (cognitive networks of relationships connecting geometric concepts and processes in specific patterns) that include visual/imagistic and verbal declarative knowledge (“knowing what”) about shapes. Typical “geometry deprived” (Fuys et al., 1988) young children have a number of imagistic schemes for triangle; for example, an equilateral triangle and a right triangle, both with a horizontal base (Hershkowitz et al., 1990; Vinner & Hershkowitz, 1980). These prototype schemes are not absolutely rigid, but they have constraints. For example, the more the lengths of the legs of a right triangle differ in length, the less likely it will be assimilated to that prototype. Such prototypes can be thought of as having multivariate distributions of possible values (e.g., for the relationships between the side lengths and for degree of the base’s rotation from the horizontal) in which the nearer to the mode of the distribution the perceived figure is (equal for the side lengths and 0° rotation of the base for this example), the more likely it will be assimilated to that prototype (parallel distributed processing, or PDP, networks model this type of system, see Clements & Battista, 1992; McClelland, Rumelhart, & the PDP Research Group, 1986). These children’s verbal declarative knowledge may include the name of the triangle and a few statements of components, such as “three sides” and possibly “three corners.” These statements, however, are not constrained further; for example, there are few limitations placed on the nature of these sides (e.g., they might be curved) and corners.

In addition, research supports previous claims (Clements & Battista, 1992) that a prerepresentational level (0) exists before van Hiele Level 1 (visual level). Children who cannot reliably distinguish circles, triangles, and squares from nonexemplars of those classes should be classified as prerepresentational; those who are learning to do so should be considered in transition to, rather than at, the visual level. Children at this level are just starting to form schemes for the shapes. These early, unconscious schemes perform pattern matching through feature analysis (Anderson, 1985; E. J. Gibson, Gibson, Pick, & Osser, 1962), even though the objects usually form undifferentiated, cohesive units in children’s experiences (c.f. Smith, 1989). For example, nascent schemes may ascertain the presence of the features of closed and “rounded” to match circles, “pointy” features to match triangles, four near-equal sides with approximately right angles to match squares, and approximate parallelism of opposite “long” sides to match rectangles.

Such prototypes match many forms, so there is a wide acceptance. With repeated exposure to exemplars in the culture, these prototypes grow stronger; to the extent these exemplars are limited (e.g., mostly equilateral for many shape categories), acceptance decreases radically, leading to the child’s rejection of both distractors and variants that do not closely match the visual prototype, which may have rigid
constraints regarding aspect ratio, skewness, and orientation. This rejection may be particularly noticeable in situations in which shapes cannot be manipulated. Nascent declarative knowledge, while developing, does not gain transcendence in the scheme. Eventually, exposure to a wider variety of examples and a strengthening of declarative knowledge (“3 straight sides”) leads to a wider acceptance of varieties of geometric figures while rejecting nonmembers of the class. Each type of knowledge increasingly constrains the other.

Certain conditions, such as a request to explain one’s decision, may prompt children, even at early levels, to abstract, attend to, and describe certain features (“this is pointy”). They might also consciously attend to a subset of the shape’s visual characteristics and use such a subset to identify geometric shapes. Thus, their descriptions of shapes may include a variety of terms and attributes (Clements & Battista, in press; Lehrer et al., 1998). However, these are “centrations” in the Piagetian sense, not integrated with other components of the shape. Again, this supports a connectionist model’s recognition of both comparison-to-prototype and attention-to-attributes in young children’s geometric categorization (Clements et al., 1999; see also Lehrer et al., 1998). Conditions also affect how such features and prototype matching are processed. For example, young children’s overall acceptance rate of both exemplars and nonexemplars increases with the inclusion of palpable distractors (Hannibal & Clements, 2000).

Thus, even at a very early age children attend to some attribute of their imagistic scheme, even if it is only its "prickliness" for triangles (c.f. Lehrer et al., 1998). Then, as the scheme becomes better formed (and as the child gains the ability and predisposition to give increased selective attention to single dimensions when comparing objects, Smith, 1989), the child is able to discern more attributes (both defining and nondefining) that s/he uses to construct her or his definition of a triangle. These schemes are consistent with the Piagetian tradition of the construction of geometric objects on the “perceptual plane” before a reconstruction on the “representational” or “imaginal plane” (Piaget & Inhelder, 1967). They can match and identify many shapes (most reliably, prototypical forms determined by both biology and culture). However, they often attend only to a proper subset of a shape’s visual characteristics and are unable to identify many common shapes. In tactile contexts, they can distinguish between figures that are curvilinear (e.g., circle) and those that are rectilinear (e.g., a square) but not among figures within those classes. Even in visual contexts, they may not be able to construct an image of shapes, or a re-presentation of the image. They are unable to rotate shapes and place them into part-whole relationships (G. Wheatley & Cobb, 1990). Thus, before level 1, students lack the ability to construct and manipulate visual images of geometric figures.

Later in development, additional visual-spatial elements, such as the right angles of squares, are incorporated into these schemes and thus traditional prototypes may be produced. Further, older children can attend to these features separately, whereas younger children are not able or predisposed to focus on single features analytically (Smith, 1989). Therefore, younger children can produce a prototype in identifying rectangles without necessarily attending to the components or specific features that constitute these prototypes. For children of all ages, the prototypes may be
overgeneralized or undergeneralized compared to mathematical categorization, of course, depending on the exemplars and nonexemplars and teaching acts children experience. Also, progress to strong level 1 and eventually to level 2 understanding is protracted. For example, primary grades continue to apply many different types of cognitive actions to shapes, from detection of features like fat or thin, to comparison to prototypical forms, to the action-based embodiment of pushing or pulling on one form to transform it into another (Clements & Battista, in press; Lehrer et al., 1998).

In summary, young children operating at levels 0 and 1 show evidence of recognition of components and attributes of shapes, although these features may not be clearly defined (e.g., sides and corners). Some children operating at level 1 appear to use both matching to a visual prototype (via feature analysis) and reasoning about components and properties to solve these selection tasks. Thus, level 1 geometric thinking as proposed by the van Hieles is more syncretic than visual, as Clements (1992) suggested. That is, this level is a synthesis of verbal declarative and imagistic knowledge, each interacting with and enhancing the other. We therefore suggest the term syncretic level, rather than visual level, signifying a global combination without analysis (e.g., analysis of the specific components and properties of figures). At the syncretic level, children more easily use declarative knowledge to explain why a particular figure is not a member of a class, because the contrast between the figure and the visual prototype provokes descriptions of differences (S. Gibson, 1985). Children making the transition to the next level sometimes experience conflict between the two parts of the combination (prototype matching vs. component and property analysis), leading to mathematically incorrect and inconsistent task performance. For example, many young children call a figure a square because it “just looked like one,” a typical holistic, visual response. However, some attend to relevant attributes; for them, a square had “four sides the same and four points”. Because they had not yet abstracted perpendicularity as an additional relevant and critical attribute, some accept certain rhombi as squares. That is, even if their prototype has features of perpendicularity (or aspect ratio near 1), young children base judgments on similarity (i.e., near perpendicularity) rather than on identity (perpendicularity), and therefore they accept shapes that are “close enough” (Smith, 1989). The young child’s neglect of such relevant (identity) attributes or reliance on irrelevant attributes leads to categorization errors. This is consistent with early findings that preschoolers show a slow development of skills, sudden insight, and regression (Fuson & Murray, 1978).

Mervis and Rosch (1981) theorized that generalizations based on similarity to highly representative exemplars will be the most accurate. This theory would account for the higher number of correct categorizations by those children who appeared to be making categorization decisions on the basis of comparison to a visual prototype without attention to irrelevant attributes. Finally, strong feature-based schemes and integrated declarative knowledge, along with other visual skills, may be necessary for high performance, especially in complex, embedded configurations. To form useful declarative knowledge, especially robust knowledge supporting transition to Level 2, children must construct and consciously attend to the components and properties of geometric shapes as cognitive objects (a learning process that requires mediation and is probably aided by physical construction tasks using manipulatives as well as reflection often prompted by discussion).
Such a theory predicts that children are developing stronger imagistic prototypes and gradually gaining verbal declarative knowledge. Those figures that are more symmetric and have fewer possible imagistic prototypes (circles and squares) are more amenable to the development of imagistic prototypes and thus show a straightforward improvement of identification accuracy. Rectangles and triangles have more possible prototypes. Rectangle identification may improve only over substantial periods of time. Similarly, shapes such as triangles, the least definable by imagistic prototypes discussed here, may show complex patterns of development as the schemes widens to accept more forms, over-widens, and then must be further constrained.

Children’s variegated responses (some visual, some verbal declarative) may be another manifestation of this syncretic level. Further, they substantiate Clements’ (1992) claim that geometric levels of thinking coexist, as previously discussed. Progress through such levels is determined more by social influences, specifically instruction, than by age-linked development. Although each higher level builds on the knowledge that constitutes lower levels, the nature of the levels does not preclude the instantiation and application of earlier levels in certain contexts (not necessarily limited to especially demanding or stressful contexts). For each level, there exists a probability for evocation for each of numerous different sets of circumstances, but this process is codetermined by conscious metacognitive control, control that increases as one moves up through the levels, so people have increasing choice to override the default probabilities. The use of different levels is environmentally adaptive; thus, the adjective higher should be understood as a higher level of abstraction and generality, without implying either inherent superiority or the abandonment of lower levels as a consequence of the development of higher levels of thinking. Nevertheless, the levels would represent veridical qualitative changes in behavior, especially the construction of mathematical representations (i.e., construction of geometric objects) from action.

To repeat a point: Geometric knowledge at every level maintains nonverbal, imagistic components; that is, every mental geometric object includes one or more image schemes, recurrent, dynamic patterns of kinesthetic and visual actions (Johnson, 1987). Imagistic knowledge is not left untransformed and merely “pushed into the background” by higher levels of thinking. Imagery has a number of psychological layers, from more primitive to more sophisticated (each connected to a different level of geometric thinking), which play different (but always critical) roles in thinking depending on which layer is activated. Even at the highest levels, geometric relationships are intertwined with images, though these may be abstract images.

Thus, images change over development. The essence of level 2 thinking, for example, lies in the integration and synthesis of properties of shapes, not merely in their recognition. Children at this level have transcended the perceptual and have constructed the properties as singular mental geometric objects that can be acted upon, not merely as descriptions of visual perceptions or images (cf. Steffe & Cobb, 1988). Ideally, however, these objects are not solely “words or pictures” (Davis, 1984), but a synthesis of verbal declarative and rich imagistic knowledge, each interacting with and supporting the other. The question, therefore, should not be whether geometric thinking is visual or not visual, but rather, whether imagery is limited to unanalyzed,
global visual patterns or includes flexible, dynamic, abstract, manipulable imagistic knowledge (Clements et al., 1999).

Figures 7 and 8 illustrate two contrasting conceptualizations of geometric levels of thinking.

Figure 7. Hypothesized linear view of the levels of geometric thinking

Note: In this view, each level ripens to fruition, engenders the beginning of the next level—which incorporates and subordinates the earlier level—and finally fades away.
Figure 8. Hypothesized synthetic view of the levels of geometric thinking

Note: In the synergistic view espoused here, types of knowledge develop simultaneously. While syncretic knowledge is dominant in the early years (darker shading indicates dominance of a particular level of thinking), descriptive knowledge is present and interacts with it, though weakly (symbolized by the small double arrow at the left). Syncretic knowledge, descriptive verbal knowledge, and, to a lesser extent initially, abstract symbolic knowledge grow simultaneously, as do their connections. When abstract knowledge begins ascendance, connections among all types are established and strengthened (symbolized by thicker arrows). Indicated only by the shading are the unconscious probabilities of instantiation associated with each level; in a related vein, but not illustrated, are the executive processes that also develop over time, serving to integrate these types of reasoning and, importantly, to determine which level of reasoning will be applied to a particular situation or task.

Considering educational practice, it is important to note that this is educational, not merely maturational, growth. If the examples and nonexamples children experience are rigid, so will be their concepts. Many children learn to accept only isosceles triangles. Others learn richer concepts, even at a young age. One of the youngest 3-year-olds in our research scored higher than every 6-year-old. Of course, it is always important to get our language “straight.” Many of the 4-year-olds in our work stated that they distinguished triangles by “three points and three sides.” Half of these children, however, were not sure what a “point” or “side” was.
Further, although appearances usually dominate children’s decisions, they are also learning and sometimes using verbal knowledge. Using such verbal knowledge accurately takes time and can initially appear as a “setback.” Children may initially say a square has “four sides the same and four points.” Because they have yet to learn about perpendicularity, some accept any rhombus as a square. Their own description convinces them even though they feel conflicted about the “look” of this “new square.” Eventually, however, this conflict can be beneficial, as they come to understand more properties of squares. We should provide varied examples and nonexamples to help children understand attributes of shapes that are mathematically relevant as well as those (orientation, size) that are not. Therefore, examples of triangles and rectangles should include a wider variety of shapes, including “long,” “skinny” and “fat” examples.

Spatial Thinking


Spatial Orientation: Maps and Navigation

Spatial orientation—knowing the shape of one’s environment—represents a domain of early cognitive strength for young children. It is probably a “core domain”—a “built-in” area of knowledge that includes the ability to actively and selectively seek out pertinent information and certain interpretations of ambiguous information (Gelman & Williams, 1997). Toddlers, for example, eschew other cues and instead use geometric information about the overall shape of their environment to solve location tasks.

Spatial orientation is knowing where you are and how to get around in the world; that is, understanding and operating on relationships between different positions in space, especially with respect to your own position. Young children learn practical navigation early—as all adults responsible for their care will attest. Channeling that experience is valuable. For example, when nursery-school children tutor others in guided environments, they build geometrical concepts (Filippaki & Papamichael, 1997).

Young children can mathematize their experiences with navigation. They can use and create simple maps and begin to build mental representations of their spatial environments. This is illustrated in 3-year-olds’ building of simple, but meaningful maps with landscape toys such as houses, cars, and trees (Blaut & Stea, 1974); however, we know less about what specific abilities and strategies they use to do so. For example, kindergarten children making models of their classroom cluster furniture correctly (e.g., they put the furniture for a dramatic play center together), but may not relate the clusters to each other (Siegel & Schadler, 1977). Also unclear is what kind of “mental
maps” young children possess. Some researchers believe that people first learn to navigate only by noticing landmarks, then by routes, or connected series of landmarks, then by scaled routes, and finally by putting many routes and locations into a kind of “mental map.” Only older preschoolers learn scaled routes for familiar paths; that is, they know about the relative distances between landmarks (Anooshian, Pascal, & McCreath, 1984). Even young children, however, can put different locations along a route into some relationship, at least in certain situations. For example, they can point to one location from another even though they never walked a path that connected the two (Uttal & Wellman, 1989). A significant proportion (40%) of 4-year-olds can not only identify that a direct and indirect route to a given location are not the same distance, but can explain why the direct route was shorter (Fabricius & Wellman, 1993).

Developing spatial orientation competencies, and eventually understanding maps, is a long-term process. Children slowly develop many different ways to represent the locations of objects in space. Infants associate objects as being near a person such as a parent (Presson & Somerville, 1985), but cannot associate objects to distance landmarks. Toddlers and 3-year-olds can place objects in pre-specified locations near distant landmarks, but “lose” locations that are not specified ahead of time once they move. Children as young as 3.5 years were able, like adults, to accurately walk along a path that replicated the route between their seat and the teacher’s desk in their preschool classroom (Rieser, Garing, & Young, 1994). They can build imagery of locations and use it, but they must physically move to show their competence. So, they may be able to form simple frameworks, such as the shape of the arrangement of several objects, that has to include their own location. With no landmarks, even 4-year-olds make mistakes (Huttenlocher & Newcombe, 1984). Kindergartners build local frameworks that are less dependent on their own position. They still rely, however, on relational cues such as being close to a boundary. By third grade, children can use larger, encompassing frameworks that include the observer of the situation.

Neither children nor adults actually have “maps in their heads”—that is, their “mental maps” are not like a mental picture of a paper map. Instead, they are filled with private knowledge and idiosyncrasies and actually consist of many kinds of ideas and processes. These may be organized into several frames of reference. The younger the child, the more loosely linked these representations are. These representations are spatial more than visual. Blind children are aware of spatial relationships by age 2, and by 3 begin to learn about spatial characteristics of certain visual language (Landau, 1988).

What about physical maps? We have seen that 3-year-olds have some capabilities building simple “maps.” There are many individual differences in such abilities. In one study, most preschoolers rebuilt a room better using real furniture than toy models. For some children, however, the difference was slight. Others placed real furniture correctly, but grouped the toy models only around the perimeter. Some children placed the models and real furniture randomly, showing few capabilities (Liben, 1988). Even children with similar mental representations may produce quite different maps due to differences in drawing and map-building skills (Uttal & Wellman, 1989).

Most children can learn from maps. For example, 4 to 7 year-olds had to learn a route through a playhouse with six rooms. Children who examined a map beforehand
learned a route more quickly than those who did not (Uttal & Wellman, 1989). Similarly, 5-6 year olds can use maps to navigate their way out of a cave (Jovignot, 1995). As with adults, then, children learn layouts better from maps than from navigation alone. Even preschoolers know that a map represents space (Liben & Yekel, 1996). More than 6 or 7-year-olds, however, they have trouble knowing where they are in the space. Therefore, they have difficulty using information available from the map relevant to their own position (Uttal & Wellman, 1989). Preschoolers, like older people, could preserve the configuration of objects when reconstructing a room depicted on a map. However, preschoolers placed objects far from correct locations and performed worse with asymmetric than symmetric configurations (Uttal, 1996). They have difficulty aligning maps to the referent space (Liben & Yekel, 1996). They may understand that symbols on maps represent objects but have limited understanding of the geometric correspondence between maps and the referent space; both understandings are developing, but have far to go, by the end of the preschool years (Liben & Yekel, 1996). By the primary grades, most children are able to draw simple sketch-maps of the area around their home from memory. They also can recognize features on aerial photographs and large-scale plans of the same area (Boardman, 1990).

What accounts for differences and age-related changes? Maturation and development are significant. Children need mental processing capacity to update directions and location. The older they get, the more spatial memories they can store and transformations they can perform. Such increase in processing capacity, along with general experience, determines how a space is represented more than the amount of experience with the particular space (Anooshian et al., 1984). Both general development and learning are important. Instruction on spatial ability, symbolization, and metacognitive skills (consciously self-regulated map reading behavior through strategic map referral) can increase 4- to 6-year-olds’ competence with reading route maps, although it does not overcome age-related differences (Frank, 1987).

Though young children possess impressive initial abilities, they have much to learn about maps. For example, preschoolers recognized roads on a map, but suggested that the tennis courts were doors (Liben & Downs, 1989)! In addition, older students are not competent users of maps. School experiences fail to connect map skills with other curriculum areas, such as mathematics (Muir & Cheek, 1986).

Fundamental is the connection of primary to secondary uses of maps (Presson, 1987). Even young children form primary, direct relations to spaces on maps. They must grow in their ability to treat the spatial relations as separate from their immediate environment. These secondary meanings require people to take the perspective of an abstract frame of reference (“as if you were there”) that conflicts with the primary meaning. You no longer imagine yourself “inside,” but rather must see yourself at a distance, or “outside,” the information. Such meanings of maps challenge people into adulthood, especially when the map is not aligned with the part of the world it represents (Uttal & Wellman, 1989). Using oblique maps (e.g., tables are show with legs) aids preschoolers’ subsequent performance on plan (“bird’s-eye view”) maps (Liben & Yekel, 1996). However, these must not be overly simple iconic picture maps, but must challenge children to use geometric correspondences. Adults need to connect the abstract and concrete meanings of map symbols. Similarly, many of young children’s
difficulties do not reflect misunderstanding about space, but the conflict between such concrete and abstract frames of reference. In summary, children (a) develop abilities to build relationships among objects in space, (b) extend the size of that space, and (c) link primary and secondary meanings and uses of spatial information.

These findings re-emphasize that we must be careful how we interpret the phrase “mental map.” Spatial information may be different when it is garnered from primary and secondary sources...such as maps.

What about the mathematics of maps? Developing children’s ability to make and use mental maps is important, and so is developing geometric ideas from experiences with maps. We should go beyond teaching isolated “map skills” and geography to engage in actual mapping, surveying, drawing, and measuring in local environments (Bishop, 1983). Such activities can begin in the early years.

Our goal is for children to both read and make maps meaningfully. In both of these endeavors, four basic questions arise: Direction—which way?, distance—how far?, location—where?, and identification—what objects? To answer these questions, students need to develop a variety of skills. Children must learn to deal with mapping processes of abstraction, generalization, and symbolization. Some map symbols are icons, such as an airplane for an airport, but others are more abstract, such as circles for cities. Children might first build with objects such as model buildings, then draw pictures of the objects’ arrangements, then use maps that are “miniaturizations” and those that use abstract symbols. Some symbols may be beneficial even to young children. Over reliance on literal pictures and icons may hinder understanding of maps, leading children to believe, for example, that certain actual roads are red (Downs, Liben, & Daggs, 1988).

Similarly, children need to develop more sophisticated ideas about direction and location. Young children should master environmental directions, such as above, over, and behind. They should develop navigation ideas, such as left, right, and front, and global directions such as north, east, west, and south, from these beginnings. Perspective and direction are particularly important regarding the alignment of the map with the world. Some children of any age will find it difficult to use a map that is not so aligned. Teachers should introduce such situations gradually and perhaps only when necessary.

Young children can learn to relate various reference frames, which brings us to the notion of coordinates. The Piagetian position is that coordinate frameworks are analogical to a container made up of a network of sites or positions (Piaget & Inhelder, 1967). Objects within this container may be mobile, but the positions are stationary. From the simultaneous organization of all possible positions in three dimensions emerges the coordinate system. This involves the gradual replacement of relations of order and distance between objects with similar relations between the positions themselves. The space is “emptied of objects.” Thus, intuition of space is not an innate apprehension of the properties of objects, but a system of relationships borne in actions performed on these objects.
In this arena, as in others, we see there is a long developmental process, but some early competencies on which to build. For example, very young children can orient a horizontal or vertical line in space (Rosser, Horan, Mattson, & Mazzeo, 1984). Similarly, 4- to 6-year-old children can extrapolate lines from positions on both axes and determine where they intersect, (b) are equally successful going from point to coordinate as going from coordinate to point, and (c) extrapolate as well with or without grid lines (Somerville & Bryant, 1985). Piagetian theory seems correct in postulating that the coordination of relations develops after such early abilities. Young students fail on double-axis orientation tasks even when misleading perceptual cues are eliminated (Rosser, Horan et al., 1984). Similarly, the greatest difficulty is coordinating two extrapolations, which has its developmental origins at the 3- to 4-year-old level, with the ability to extrapolate those lines developing as much as a year earlier (Somerville, Bryant, Mazzocco, & Johnson, 1987). These results suggest an initial inability to utilize a conceptual coordinate system as an organizing spatial framework (Liben & Yekel, 1996). Some 4-year-olds can use a coordinate reference system, whereas most 6-year-olds can (Blades & Spencer, 1989). However, four-year-olds can coordinating dimensions if the task is set in a meaningful context in which the orthogonal dimensions are cued by the line of sights of imaginary people (Bremner, Andreasen, Kendall, & Adams, 1993).

Coordinate of coordinates is not limited to two orthogonal dimensions. Children as young as five years can metrically represent spatial information in a polar coordinate task, using the same two dimensions as adults radius and angle, although children do not use categorizations of those dimensions until age 9 (Sandberg & Huttenlocher, 1996).

In summary, even young children can use coordinates that adults provide for them. However, when facing traditional tasks, they and their older peers may not yet be able or predisposed to spontaneously make and use coordinates for themselves.

Computer activities can facilitate learning of navigational and map skills. Young children can abstract and generalize directions and other map concepts working with the Logo turtle (Borer, 1993; Clements, Battista, Sarama, Swaminathan, & McMillen, 1997; Clements & Meredith, 1994; Goodrow, Clements, Battista, Sarama, & Akers, 1997; Kull, 1986; Try, 1989; Watson, Lange, & Brinkley, 1992; Weaver, 1991) although results and not guaranteed (Howell, Scott, & Diamond, 1987). The interface must be appropriate and activities must be well planned (Watson & Brinkley, 1990/91). Giving the turtle directions such as forward 10 steps, right turn, forward 5 steps, they learn orientation, direction, and perspective concepts, among others. Walking paths and then recreating those paths on the computer help them abstract, generalize, and symbolize their experiences navigating. For example, one kindergartner abstracted the geometric notion of “path” saying, “A path is like the trail a bug leaves after it walks through purple paint.”

Logo can also control a floor turtle robot, which may have special benefits for certain populations. For example, blind and partially sighted children using a computer-guided floor turtle developed spatial concepts such as right and left and accurate facing movements (Gay, 1989).
Other simple (non-Logo) navigational programs may have similar benefits. For example, using such software (with on-screen navigation) has shown to increase kindergartners’ understanding of the concepts of left and right (Carlson & White, 1998).

Coordinate-based games on computers can help older children learning location ideas (Clements et al., 1998). When children enter a coordinate to move an object but it goes to a different location, the feedback is natural, meaningful, nonevaluative, and so particularly helpful.

Many people believe that maps are “transparent”—that anyone can “see through” the map immediately to the world that it represents. This is not true. Clear evidence for this is found in students’ misinterpretations of maps. For example, some believe that roads colored red on a map are red in the real world; others may believe that a river is a road or that a pictured road is not a road because “it’s too narrow for two cars to go on.” Even adults do not really understand maps. They believe that maps are simply miniaturizations of the world.

Students should see that maps do not show what “is”; rather, they communicate a certain “view.” They should understand that maps are different ways of viewing the world, in a way that is comparable to different artistic interpretations. They are models that help us see what we often can not see in the real world. Different models show the world in different ways—“This is what the world would look like if….” In the long term, students should understand that maps let us “see” aspects of the world that we could not see without them. They allow us to inspect and transform the larger world in new ways. They empower us in perceiving relationships about the world that we would not have noticed without the structural characteristics of the map.

Spatial Visualization and Imagery

Spatial visualization the ability to generate and manipulate images. Kosslyn (1983) defines four classes of image processes: generating an image, inspecting an image to answer questions about it, maintaining an image in the service of some other mental operation, and transforming and operating on an image. Thus, spatial visualization involves understanding and performing imagined movements of two- and three-dimensional objects. To do this, you need to be able to create a mental image and manipulate it. An image is not a “picture in the head.” It is more abstract, more malleable, and less crisp than a picture. It is often segmented into parts. As we saw, some images can cause difficulties, especially if they are too inflexible, vague, or filled with irrelevant details.

People’s first images are static. They can be mentally re-created, and even examined, but not transformed. For example, you might attempt to think of a group of people around a table. In contrast, dynamic images can be transformed. For example, you might mentally “move” the image of one shape (such as a book) to another place (such as a bookcase, to see if it will fit). In mathematics, you might mentally move (slide) and rotate an image of one shape to compare that shape to another one. Piaget argued that most children cannot perform full dynamic motions of images until the primary grades (Piaget & Inhelder, 1967, 1971). However, preschool children show initial transformational abilities, as we discussed in previous sections.
Spatial Sense

Spatial sense includes two main spatial abilities: spatial orientation and spatial visualization and imagery. Other important knowledge includes how to represent ideas in drawing and how and when you can use such abilities.

This view clears up some confusion regarding the role of spatial sense in mathematics thinking. “Visual thinking” and “visual strategies” are not the same as spatial sense. Spatial sense as we describe it—all the abilities we use in “making our way” in the spatial sphere—is related to mathematical competencies (Brown & Wheatley, 1989; Clements & Battista, 1992; E. Fennema & Carpenter, 1981; G. H. Wheatley, Brown, & Solano, 1994).

Visual thinking, as in the initial levels of geometric thinking, is thinking that is tied down to limited, surface-level, visual ideas. Children move beyond that kind of visual thinking as they learn to manipulate dynamic images, as they enrich their store of images for shapes, and as they connect their spatial knowledge to verbal, analytic knowledge. Teachers might encourage children to describe why a shape does or does not belong to a shape category.

Early Childhood Geometry: Implications for Scope and Sequence and Instruction

These findings have substantial implications for curriculum and instruction in early childhood education. This section describes these implications. Given that this research review was generated to inform the Building Blocks project, the attached Early Childhood Geometry Curriculum Goals in the Appendix represents our interpretation of the research and implications for such curriculum specifications.

Geometric Figures

The belief that children are geometric tabula rasa is untenable; preschool children exhibit working knowledge of shapes. Instruction should build on this knowledge and move beyond it. Unfortunately, present curriculum and practice (including the home and preschool) rarely does so. Very young children can learn rich concepts about shape if provided with varied examples and nonexamples, discussions about shapes and their characteristics, and interesting tasks. Let us consider each of these in more depth.

Research indicates that curricula should ensure that children experience many different examples of a type of shape. For example, Figure 3 shows a rich variety of triangles and distractors that would be sure to generate discussion. We should also show nonexamples that, when compared to similar examples, help focus attention on the critical attributes.

Discussions should encourage children’s descriptions while encouraging the development of precise language. Early talk can clarify the meanings of terms. With such clarification, children can learn to explain why a shape belongs to a certain category—“It has three straight sides.” Eventually, they can internalize such arguments;
for example, thinking, “It is a weird, long, triangle, but it has three straight sides!” Finding and identifying shapes by feeling is one useful activity (see Fig. 9).

Figure 9

We should encourage children to describe why a figure belongs or does not belong to a shape category. Visual (prototype-based) descriptions should, of course, be expected and accepted, but attribute and property responses should also be encouraged. They may initially appear spontaneously for shapes with stronger and fewer prototypes (e.g., circle, square). They should be especially encouraged for those shape categories with more possible prototypes, such as triangles. In all cases, the traditional, single-prototype approach must be extended.

Early childhood curricula traditionally introduce shapes in four basic level categories: circle, square, triangle, and rectangle. The idea that a square is not a rectangle is rooted by age five (Clements et al., 1999; Hannibal & Clements, 2000). It is time to re-think our presentation of squares as an isolated set. If we try to teach young children that “squares are rectangles,” especially through direct telling, confusion is likely. If, on the other hand, we continue to teach “squares” and “rectangles” as two separate groups, we will block children’s transition to more flexible categorical thinking.

In our study (Clements et al., 1999), 4-year-olds were more likely to accept the squares as rectangles, possibly because they were less predisposed (because their prototype of rectangles was less distinguished from that of squares) or able to judge equality of all sides. Although the squares were included in the rectangle-recognition task (by the original task designers) to assess hierarchical inclusion, we did not expect or find such thinking in these young children. Their responses do show, however, that the path to such hierarchical thinking is a complex and twisting one with changes at
several levels. This again raises the question of whether the strictly visual prototype approach to teaching geometric shapes is a necessary prerequisite to more flexible categorical thinking or a detriment to the early development of such thinking. Kay (1987) provided first graders with instruction that (a) began with the more general case, quadrilaterals, proceeded to rectangles, and then to squares; (b) addressed the relevant characteristics of each class and the hierarchical relationships among classes; and (c) used terms embodying these relationships (“square-rectangle”). At the end of instruction, most students identified characteristics of quadrilaterals, rectangles, and squares, and about half identified hierarchical relationships among these classes, although none had done so previously. Although the depth of these first-graders’ understanding (especially of hierarchical relations) and the generalizations made on the basis of the empirical results must be questioned (Clements & Battista, 1992), so too should we question the wisdom of the traditional, prototype-only approach, which may lay groundwork that must be overturned to develop hierarchical thinking.

Probably the best approach is to present many examples of squares and rectangles, varying orientation, size, and so forth, including squares as examples of rectangles. If children say “that’s a square,” teachers might respond that it is a square, which is a special type of rectangle, and they might try double-naming (“it’s a square-rectangle”). Older children can discuss “general” categories, such as quadrilaterals and triangles, counting the sides of various figures to choose their category. Also, teachers might encourage them to describe why a figure belongs or does not belong to a shape category. Then, teachers can say that because a triangle has all equal sides, it is a special type of triangle, called an equilateral triangle. Children might also “test” right angles on rectangles with a “right angle checker” (angle as turn is addressed in the following section).

Logo microworlds can be evocative in generating thinking about squares and rectangles for young children. In one large study (Clements & Battista, in press), some kindergartners formed their own concept (e.g., “it’s a square rectangle”) in response to their work with the microworlds. This concept was applied only in certain situations: Squares were still squares, and rectangles, rectangles, unless you formed a square while working with procedures—on the computer or in drawing—that were designed to produce rectangles. The concept was strongly visual in nature, and no logical classification per se, such as class inclusion processes, should be inferred. The creation, application, and discussion of the concept, however, was arguably a valuable intellectual exercise.

Also, children can and should discuss the parts and attributes of shapes. Activities that promote such reflection and discussion include building shapes from components. For example, children might build squares and other polygons with toothpicks and marshmallows or other objects (see Fig. 10). They might also form shapes with their bodies, either singly or with their friends (see Fig. 11). Again, computer-based shape manipulation and navigation (including turtle geometry and simpler) environments can help mathematize these experiences.

Figure 10
Shape concepts begin forming in the preschool years and stabilize as early as age 6 (Gagatsis & Patronis, 1990; Hannibal & Clements, 2000). It is therefore critical that children be provided better opportunities to learn about geometric figures between 3 and 6 years of age. Curricula should develop early ideas aggressively, so that by the end of grade 2 children can identify a wide range of examples and non-examples of a wide range of geometric figures; classify, describe, draw, and visualize shapes; and describe and compare shapes based on their attributes.

Figure 11
Angle and Turn

Students struggle with angles. To understand angles, they must understand the various aspects of the angle concept, overcome difficulties with orientation, discriminate angles as critical parts of geometric figures, and construct and represent the idea of turns. Furthermore, they must construct a high level of integration between these aspects. Some argue that this difficult concept should not be a component of the early childhood mathematics curriculum. In contrast, the research reviewed here indicates that children do have initial competencies in the domain of turns and angles, and that the long developmental process is best begun in the early and elementary classrooms, as children deal with corners of figures, comparing angle size, and turns. Computers can help children quantify turns and angles (Fig. 12).

Figure 12

The role parallelism and perpendicularity should play are less clear. It may be that embedded in an overall approach to angle, turn, shape, and spatial structure, these ideas can be successfully nurtured, but whether they should be abstracted and discussed as separate concepts (e.g., lessons on perpendicular lines) requires additional research-based curriculum development.

3-D Figures

As with 2-D figures, children need more and richer experiences with solids. Research indicates that construction activities involving nets (foldout shapes of solids) may help students learn to discriminate between 2-D and 3-D figures. Practitioners and
Curriculum developers report success providing many other experiences; we need research to better describe, explain, and develop these approaches.

**Congruence, Symmetry, and Transformations**

Beginning as early as 4 years of age, children can create and use strategies for judging whether two figures are “the same shape.” In the preK-grade 2 range, they can develop sophisticated and accurate mathematical procedures for determining congruence.

There is mixed evidence regarding young children’s ability with geometric motions. PreK-K, and even grade 1-2 children, may be limited in their ability to mentally transform shapes, although there is evidence that even these sophisticated processes are achievable. Further, they can learn to perform rotations on objects (physical or virtual), and a rich curriculum, enhanced by such manipulatives and computer tools, may reveal that knowledge and mental processes are valid educational goals for most young children.

Similarity is a surprising area of competence for young children. Young children can identify similar shapes in certain situations and use computers to create similar shapes. First and second graders can identify similar shapes and use scaling transformations to check their predictions.

Symmetry is also an area of strength (see Fig. 13). There is undeveloped potential in generating curricula that seriously consider children’s intuitions, preference, and interest in symmetry.

*Figure 13. Children intuitively use symmetry in their block buildings.*
Composition and Decomposition

Preschool children move through levels in the composition and decomposition of 2-D figures. From lack of competence in composing geometric shapes, they gain abilities to combine shapes into pictures, then synthesize combinations of shapes into new shapes (composite shapes), eventually operating on and iterating those composite shapes. Few curricula challenge students to move through these levels.

Spatial Orientation: Maps and Navigation

Spatial orientation—knowing the shape of one’s environment—is perhaps even more an area of early intuitive knowledge than the domain of small shapes. Very young children know and use the shape of their environment in navigation activities, and, with guidance, can learn to mathematize this knowledge. They can learn about direction, perspective, distance, symbolization, location, and coordinates. Some studies have identified first grade as the period of most efficient learning of maps, but informal experiences in preschool and kindergarten are also beneficial, especially those that emphasize building imagery from physical movement. (See Figure 14.)
Spatial Visualization and Imagery

Even preschool and kindergarten children show initial transformational abilities in certain settings (see the “Transformations” section). All children should work on developing their ability to create, maintain, and represent mental images of mathematical objects.

Final Words

Research has clearly identified that children’s informal numerical knowledge develops through the preschool years. While not as extensively documented, there is sufficient research indicating that informal geometric knowledge similarly develops throughout the early childhood years. There is some evidence that there is little to lose, and much to gain, by fostering that development. Especially given children’s affinity toward, knowledge of, and ability to gain geometric and spatial knowledge, it would be an educational shame to allow the U.S. obsession with number (both in practice and research) to reinstantiate itself in the nascent domain of early childhood mathematics education.
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Appendix. Early Childhood Geometry Curriculum Goals and Learning Trajectories

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<tr>
<td>S1</td>
</tr>
<tr>
<td>S2</td>
</tr>
<tr>
<td>S3</td>
</tr>
</tbody>
</table>

**Analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships** (S1, S2, S3)

- recognize, name, build, draw, compare, and sort two-and three-dimensional shapes (S1, S2)

  - Identify 2-D shapes, including examples and non-examples of a variety of shapes (semi-circles, trapezoids, rhombi, hexagons, in any orientation)³
    - Match shapes, first with same size and orientation, then with different sizes and orientations⁶
  - Recognize and name prototypical circle, square, triangle, rectangle
  - Recognize and name circle, square, triangle, rectangle (i.e., in any size or orientation; varying shapes for triangles and rectangles)
  - Recognize and name a wider variety shapes (circles, squares, rectangles, triangles, semi-circles, quadrilaterals, trapezoids, rhombi, hexagons, in any orientation)

  - Classify 2-D shapes by category, including non-standard orientations and shapes
    - Use class names informally
    - Use shape class names to classify and sort, informally --> formally
    - Use class membership for shapes, based on properties

  - Visualize, describe, draw and represent 2-D shapes (includes geometric paths representing “route maps”)

| PK12 |
Identify congruent and non-congruent 2-D shapes.

*Side-matching strategy*

*Side- and angle- matching strategy*

*Use superposition and rigid transformations*

Name, describe, compare, and sort 3-D concrete objects.

**describe attributes and parts of two-and three-dimensional shapes (S2)**

Identify component parts of shape

*Identify and count sides*

*Identify turns and angles*

*Independently identify shapes in terms of their components; this is defining attribute. That is, use components to define shapes—“It has 1, 2, 3 sides…it’s a triangle.”*

Identify parallelism and perpendicularly

Describe the faces of solids as 2-D shapes and construct models of real-world objects using solids.

Identify parts of 3-D shapes, including faces, corners and edges; construct 3-D structures from 2-D representations (photographs or drawing).

**investigate and predict the results of putting together and taking apart two- and three-dimensional shapes (S3 S1)**

Compose (put together) 2-D shapes to make new shapes

*Cover an outline (or “frame”) with other shapes*

*Compose (combining shapes into new shapes, matching side lengths and angles)*

*Compose (with substitution) shapes*

Decompose (break apart) 2-D shapes to make new shapes

*Decompose simple shapes that have palpable spatial clues as to their decomposition*

*Decompose shapes using imagery that is suggested and supported by the task or environment*

*Decompose shapes flexibly using independently generated imagery*
### Locations, Directions, and Coordinates

**L1**

Mathematics can be used to precisely specify directions, routes, and locations in the world.

<table>
<thead>
<tr>
<th><strong>Specify locations and describe spatial relationships using coordinate geometry or other representational systems (L1)</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>describe, name, and interpret relative positions in space and apply ideas about relative position (L1)</strong></td>
</tr>
<tr>
<td>Understand and use ideas such as over, under, above, on, beside, next to, between.</td>
</tr>
<tr>
<td><strong>describe, name, and interpret, direction and distance in navigating space and apply ideas about direction and distance (L1)</strong></td>
</tr>
<tr>
<td>Understand the relationship between a 3-D space and a 2-D representation of that space (map), including paths as representations of movement in those spaces.</td>
</tr>
<tr>
<td><em>Places toy objects in correct relative position to make a map of the classroom</em></td>
</tr>
<tr>
<td><em>Can make and follow maps of familiar areas</em></td>
</tr>
</tbody>
</table>
find and name locations with simple relationships such as “near to” and in coordinate systems such as maps (L1)

Explore scaling and similarity

Uses coordinates to find and name locations, including double-axis orientation and coordinate labels.

*Orient a horizontal or vertical line in space*

*Extrapolate lines from positions on both axes and determine where they intersect, simple situations*

*Use coordinate labels in simple situations*

*Coordinates two axes in multiple situations*

*Plots coordinates*

---

### Transformations and Symmetry

<table>
<thead>
<tr>
<th>TS1²</th>
<th>Mathematical transformations can be used to precisely move and change shapes.</th>
</tr>
</thead>
<tbody>
<tr>
<td>TS2</td>
<td>Symmetry can be used to analyze, understand, and create shapes in geometry and art.</td>
</tr>
</tbody>
</table>

- **Apply transformations and use symmetry to analyze mathematical situations** (TS1²)

- **recognize and apply slides, flips, and turns** (TS1)
  - Perform slides, flips, and turns of 2-D shapes
    - *Trial-and-error strategy*
    - *Identify geometric motions*
    - *Visualize slides, flips, and turns of 2-D and 3-D shapes.*
    - *Predict the outcome of motions—slide, flip, and turn*
  - Apply similarity transformations and identify similar shapes.

- **recognize and create shapes that have symmetry** (TS2)
  - Identify and create 2-D shapes that have line or rotational symmetry
Visualization and Spatial Reasoning

V1: Mental images can be used to represent and manipulate shapes, directions, and locations.

V2: Objects can be represented from different points of view.

<table>
<thead>
<tr>
<th>Use visualization, spatial reasoning, and geometric modeling to solve problems (V1\textsuperscript{2} V2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>create mental images of geometric shapes using spatial memory and spatial visualization (V1)</td>
</tr>
<tr>
<td>Make and draw shapes from memory.</td>
</tr>
<tr>
<td>Create a shape from verbal directions.</td>
</tr>
<tr>
<td>recognize and represent shapes from different perspectives (V2)</td>
</tr>
<tr>
<td>relate ideas in geometry to ideas in number and measurement (V1 M1)</td>
</tr>
<tr>
<td>recognize geometric shapes and structures in the environment and specify their location (V1 S1)</td>
</tr>
<tr>
<td>Give and follow directions for moving in physical space and on a map, including understanding geometric paths as representations of that movement.</td>
</tr>
<tr>
<td>Understand that maps answer questions about direction, distance and location.</td>
</tr>
</tbody>
</table>

\textsuperscript{1} Children’s capabilities in geometry and spatial sense, especially given the sporadic curricula they usually experience, vary greatly. Ages at which children achieve capabilities, then, are dependent on teaching and learning. We have combined developmental psychology research and classroom-based research in this chart. Therefore, these represent achievable goals given appropriate educational experiences. The symbols can be interpreted as follows.

- ○ = Informal, intuitive knowledge is developing. This suggests at most informal, incidental work with the objective.
- ★ = The objective can be taught and learned at this age.
- ☀ = The objective can be taught and learned well at this age; indeed, it can be learned sufficiently well that it is considered “mastered.”
Labeled sentences are the “big topic ideas” for geometry in Figures 1 and 2, respectively. For example, S1 is the first big idea in Figure 1 for the topic of Shapes. These symbols in parentheses, such as (S1, S2), connect specific goals to the big topic ideas.

Specific goals (underline bold) correspond to a main standard in NCTM’s PSSM. As mentioned, the labels following such goals in parentheses specify which big topic idea(s) they address.

Detailed objectives correspond to specific preK-2 expectation in NCTM’s PSSM.

Indented levels provide additional specifics for detailed objectives, providing guidelines for children’s development.

The most indented level (this size font, in italics) provide specific research-based learning trajectories.