Journal for Research in Mathematics Education
1997, Vol. 28, No. 1, 70–95

Students' Development of Length Concepts in a Logo-Based Unit on Geometric Paths

Douglas H. Clements, *State University of New York at Buffalo*
Michael T. Battista, *Kent State University*
Julie Sarama, *Wayne State University*
Sudha Swaminathan, *University of Nevada at Las Vegas*
Sue McMillen, *D'Youville College*

We investigated the development of linear measure concepts within an instructional unit on paths and lengths of paths, part of a large-scale curriculum development project funded by the National Science Foundation (NSF). We also studied the role of noncomputer and computer interactions in that development. Data from paper-and-pencil assessments, interviews, and case studies were collected within the context of a pilot test of this unit with 4 third graders and field tests with 2 third-grade classrooms. Three levels of strategies for solving length problems were observed: (a) apply general strategies such as visual guessing of measures and naive guessing of numbers or arithmetic operations; (b) draw hatch marks, dots, or line segments to partition lengths to serve as perceptible units to quantify the length; (c) no physical partitioning—use an abstract unit of length, a "conceptual ruler," to project onto unsegmented objects. Those students who had connected numeric and spatial representations evinced different and more powerful problem-solving strategies in geometric situations than those who had forged fewer such connections.

There is a need for curriculum units that develop geometric knowledge and spatial sense in ways consistent with recent recommendations for reform (Clements & Battista, 1992; Kouba et al., 1988; Lindquist & Kouba, 1989; National Council of Teachers of Mathematics, 1989). Such units imply changes to both the present content and methods of teaching; therefore, there is a corresponding need and a responsibility to chart students' development of mathematical knowledge as they work in such units. We are engaged in a large-scale curriculum-development project that emphasizes meaningful mathematical problems and depth rather than mere exposure. We are particularly interested in geometry and spatial-sense units in this curriculum. One unit developed in this project, Turtle Paths, engages third-grade students in a series of combined geometric and arithmetic investigations exploring paths and

This work was supported by two National Science Foundation grants: ESI-9050210 and MDR-8954664. The first, "Investigations in Number, Data, and Space: An Elementary Mathematics Curriculum," was a cooperative project among the State University of New York at Buffalo (UB), Kent State University (KSU), and TERC. The second, "An Investigation of the Development of Elementary Children's Geometric Thinking in Computer and Noncomputer Environments," is a cooperative research project between UB and KSU. Any opinions, findings, and conclusions or recommendations expressed in this publication are those of the authors and do not necessarily reflect the views of the National Science Foundation.

the lengths of paths. We have been investigating students' learning within this unit, emphasizing their developing ideas about the measurement of length.

The basis of both the study's theoretical underpinnings and the development of the instructional unit is that students' initial representations of space are based on action, rather than on passive "copying" of sensory data (Piaget & Inhelder, 1967). An implication is that noncomputer and Logo turtle activities designed to help students abstract the notion of path—a record or tracing of the movement of a point—provide a fertile environment for developing their conceptualizations of simple two-dimensional figures and their measures. Students' actions in these environments may include activities that are both perceptual—scanning paths in the classroom and watching the turtle draw paths—and physical—walking and drawing paths and later interpreting the turtle's movement as physical motions like one's own (Clements & Battista, 1992). Having and then consciously connecting such experiences may help students build units of length by segmenting continuous motion (Steffe, 1991). (Here we use "segmenting" to mean dividing a length into parts and "partitioning" to mean dividing, or intending to divide, a length into equal-interval parts.) Previous research indicates that the computer experience can play a pivotal role in helping students forge those connections (Clements & Meredith, 1993).

On the basis of this foundation, we conjecture that combined noncomputer and Logo experiences will positively affect length measurement competencies. The emphasis on paths and movement might help students focus on intervals or line segments as units of length, instead of on points (Cannon, 1992). Further, connections between mathematical symbols and graphics may promote in students a belief in the necessity of equal-interval units, another significant conceptual advance (Petitto, 1990). In addition, these connections may help students build mental connections between numerical and geometric ideas. There is some evidence that supports these claims, though the types of items on which effects have been measured and found have been limited; for example, relative lengths of horizontal and oblique lines on a grid, or use of arithmetic in a real-world geometric setting (Clements & Battista, 1989; Clements & Battista, 1992; Noss, 1987).

Students working with Logo can manipulate units and explore transformations of unit size and number of units without the particular dexterity demands associated with measuring instruments and physical quantity. In one study, children with Logo experience were more accurate than control children in measurement tasks (Campbell, 1987). This is significant, given that students have difficulty dealing with quantities measured with different units (Carpenter & Lewis, 1976; Hiebert, 1981). The control children were more likely to underestimate distances, particularly the longest distances; have difficulty compensating for the halved unit size; and underestimate the inverse relationship between unit size and unit numeracy (Campbell, 1987).

Our goal in the present study was to investigate the development of linear-measure concepts within an instructional unit on geometric paths, including the role of noncomputer and computer interactions in that development. We adopted a multifaceted methodology, emphasizing an openness to emergent conjectures and theories (Strauss & Corbin, 1990).

METHOD

Procedure

A unit of instruction, Turtle Paths, was taught to third graders in two different situations. In the first, a pilot test, a graduate assistant taught the unit to four students in the spring of 1992. Pre- and postinterviews and case studies were conducted with each student. The second situation, a field test, involved two of the authors teaching the unit to two third-grade classes the following autumn. Data collection for the field test included pre- and postinterviews, paper-and-pencil pretests and posttests, case studies, and whole-class observation.

Participants

Participants in the pilot test were four students from a rural town—two girls, Anne and Barb, and two boys, Charles and David—all 9 years of age. They worked in same-sex pairs on off- and on-computer tasks; a case study was conducted with each student. Participants for the field test were 38 students in two heterogeneously grouped inner-city third-grade classes. Each class was a heterogeneous group representing the school population; 80% of the students qualified for Chapter 1 assistance in mathematics. One student in each class, identified by the teacher as a talkative student with good attendance, was studied intensely (i.e., case studies were conducted of Luke and Monica). When students worked in pairs, Monica's partner Nina was also observed (Luke's partner's low attendance did not allow for meaningful data collection). In addition, one of the field-test teachers identified two additional students, Oscar and Peter, as among the lowest performers in mathematics and placed them in the front of the class so that she could "keep an eye on them." The researcher who taught the activities and a researcher who observed the class as a whole observed this pair frequently.

Curriculum

The Turtle Paths unit engages third-grade students in a series of combined geometric and arithmetic investigations. The unit teaches about geometric figures (e.g., paths, rectangles, squares, and triangles); geometric processes such as measuring, turning, and visualizing; and arithmetic computation and estimation. Throughout the unit, students explore paths and the lengths of paths. The unit consists of three investigations, each involving several sessions (See Table 1).

Geo-Logo *Environment*

A modified Logo environment, *Geo-Logo* (Clements & Meredith, 1994), is an intrinsic component of the instructional unit. *Geo-Logo*'s design is based on curricular considerations and a number of implications for the learning and teaching of geometric concepts with turtle graphics (Clements & Sarama, in press). For example, although research indicates that Logo experiences can help students learn geometry, it also indicates that students often continue to use visually based, nonanalytical approaches (van Hiele, 1986). *Geo-Logo* was designed on five principles abstracted from research (Clements & Sarama, 1995): (a) encourage construction of the abstract from the visual, (b) maintain close ties between representations, (c) facilitate examination and modification of code and thus exploration of mathematical ideas, (d) encourage procedural thinking, and (e) provide freedom within constraints (e.g., turns were restricted to multiples of 90° or 30° in various activities).

Table 1
Description of the Investigations in the Turtle Path Unit

Session	Description
	Investigation 1: Paths and Lengths of Paths
1	Students walk, describe, discuss, and create paths. They give Logo commands to student-robots to specify movements that create paths. This introduces a formal geometric symbolization that is built up during the remainder of the unit.
2	In the activity, Maze Steps, students count steps in a maze (in which steps are in groups of 5) to find certain paths, such as a path that is 14 steps in length and has two corners, and play a Maze Paths game in which they roll dice and move a marker the indicated number of steps. Problems with more than one possible solution are emphasized.
3–4.	Students give movement commands to the Logo turtle to create paths in a Get the Toys game (see Figure 1). Only 90° turns are used. The computer activity is designed to promote thoughtful use of the commands (as opposed, e.g., to nonreflective trial and error). An on-screen battery's limited energy decreased with each command used.

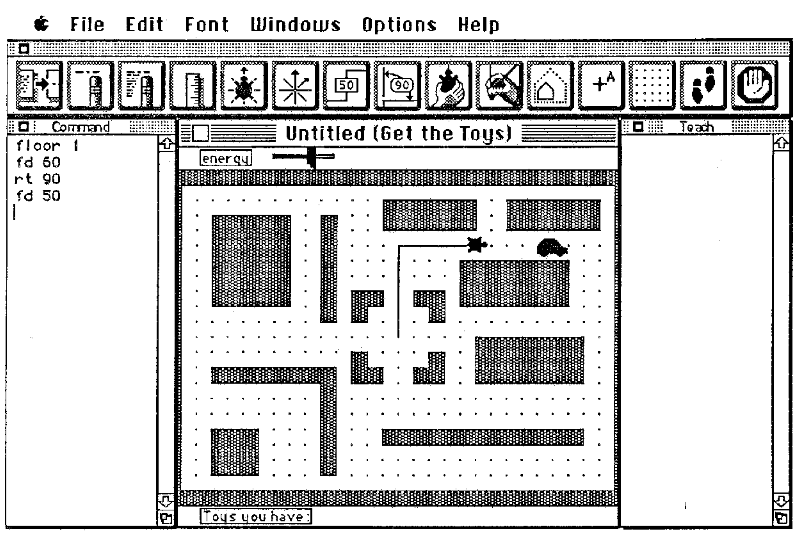


Figure 1. In the Geo-Logo game Get the Toys, students instruct the turtle to get three toys (one on each "floor") before its battery runs out of energy. Each command uses one unit of energy, regardless of the size of the input to that command. (Geo-Logo™ and Turtle Math™, a stand-alone version, copyright Douglas H. Clements and Julie Sarama Meredith. Development system copyright, Logo Computer Systems, Inc. All rights reserved.)

Session	Description
	Investigation 2: Turns in Paths
5	Students turn their bodies, discuss ways to measure turns, and learn about degrees. They are introduced to another computer game, Feed the Turtle, in which they must use their knowledge of turns to direct the turtle through channels of water (turns in this game are multiples of 30°). The turtle continues to lose energy with every command unless it eats a pair of berries, in which case its

(table continued)

Table 1—continued
Description of the Investigations in the Turtle Path Unit

Session	Description
	Investigation 2: Turns in Paths—continued
	energy is replenished. There are 7 such pairs of berries on the screen.
6	Students play the Feed the Turtle game and thus continue to estimate and measure turns. They discuss the nature of triangles and build their own descriptions of that class of geometric figures.
7	Students write Logo procedures to draw equilateral triangles.
8–9	Students find the Missing Measures (lengths and turns) to complete partially drawn paths (see Figures 5 and 6 for examples). The students must figure out that the turtle must move fd 10 (forward 10 turtle steps) to complete a side of a figure, and so on. In this and more complex problems, they must analyze geometric situations and use mental computation in a meaningful setting.
	Investigation 3: Paths With the Same Length—Isometric Exercises
10–11	Students write as many procedures as possible to draw different rectangles of a certain overall length or perimeter—200 Steps. They begin by planning off computer. They check their procedures on computer and then continue to plan and check more procedures while on computer. As with the missing lengths and turns problems, they must analyze geometric situations and use mental computation.
12–14	Students design and program the computer to draw a picture of a face, for which each part (e.g., ear, mouth) has a predetermined length or perimeter; the shape (rectangle, square, or equilateral triangle) is the students' choice.

As an example of putting these principles into practice, a critical feature of *Geo-Logo* is that students enter commands in "immediate mode" in a command window (though they can also enter procedures in the "Teach" window; see Figure 1). Any change to these commands is reflected automatically in the drawing. For example, if a student changes fd 20 to fd 30, the change is immediately reflected in a corresponding change in the geometric figure. The dynamic link between the commands and the geometry of the figure is critical; the commands in the command window always precisely reflect the geometry of the figure.

Other features include a variety of icon-based tools. Tools for writing and editing code include a tool for easy defining of procedures (the icon on the left of the tools bar in Figure 1), editing (the next two tools erase one command in the "Command" window and all commands, respectively, again with dynamic links to the geometric figure), and inspecting and changing commands (the "walking feet" is a "Step" icon, which allows students to "walk through" any sequence of commands; each command is simultaneously highlighted and executed). Tools to enhance measurement include measuring tools (the ruler measures lengths, the next two tools measure turns and angles), and labeling tools (labeling lengths and turns).

Data Collection

Case studies. One researcher sat next to and observed each case-study student throughout each session. If the student's thinking could not be ascertained through passive observation, the researcher would ask the student to think aloud or would pose specific questions.

Interviews. All four pilot students were interviewed informally following the completion of the unit; 13 students from the field test (originally 7 students were selected by each teacher; 3 average, 2 above, and 2 below; 1 student left the school in the middle of the study) were interviewed using a structured protocol on either the first or second day of the unit and then again after completion of the unit. Oscar, Peter, and the case-study students were included as interview students. Individual items in the interview are described in the results section.

Paper-and-pencil test. A short paper-and-pencil test was administered to seven field-test students following the pretreatment interview to determine its appropriateness for this age student. Students appeared to cope with and understand the items adequately; therefore, the test was administered to the whole class as a posttest. In retrospect, it was unfortunate that the test was not given to the entire class as a pretest, even though the paper-and-pencil data do not weigh heavily in the analyses. As with the interview, no total scores or statistics were computed; rather, individual items were analyzed. These are described as appropriate in the following section.

RESULTS AND DISCUSSION

Four themes emerged: concepts of length and unit of length, with an emphasis on segmenting and partitioning; combining and decomposing processes in a length context; connections between students' number and spatial schemes; and the role of the computer environment in students' development of processes and concepts regarding length.

Concept of Length and Unit of Length

Segmenting and partitioning. In each kind of situation dealing with length, at least some students drew dots or hash marks to aid in creating line segments of a certain length, assigning a measure to an already-drawn line segment, or labeling a given line segment with a given measure.

Students' constructed segments were, especially at first, usually not units. For example, when planning a rectangle off computer for the 200 Steps activity (Session 10), the pilot girls attempted to draw a segment of 73 units by counting to 70, drawing a dot while simultaneously counting out loud each multiple of 10, then adding additional equal-interval segments and counting, "seventy-one, seventy-two, seventy-three" (See Figure 2). Students also were observed stretching the intervals to meet a preconceived notion for the length of a line segment. For example, they often drew a segment and marked it while counting, but changed the scale as they neared the end of the segment so that the final count of these marks fit their preconception of the length of the line segment.

Students using this strategy tended not to establish and maintain connections among the numbers for the measures, the dots they drew to indicate a measure of length, and the shape and size of the geometric figures. As a final example, pilot students were observed marking off dots for two sides of a figure they were drawing (for the 200 Steps activity) and then labeling both sides 73 units of length, despite the noticeable difference in length of the two sides and the discrepancies in the distances between the dots on the same sides.

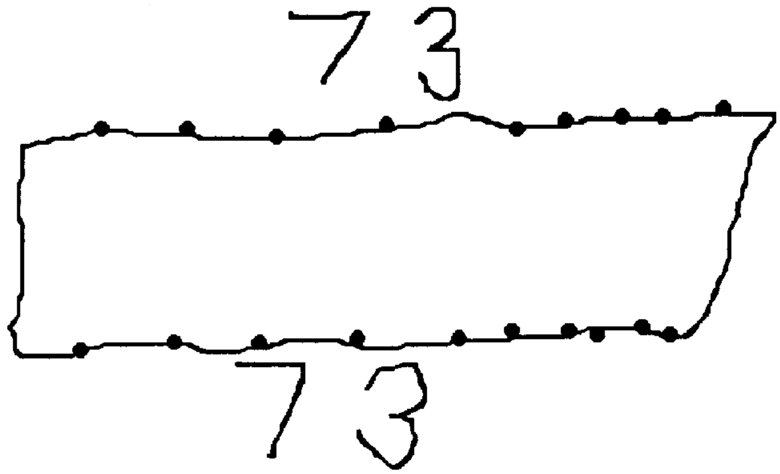


Figure 2. Rectangle of length 73 units drawn by the pilot girls in the 200 Steps activity

This need for physical marks to estimate distances was observed in other activities as well. In the Get the Toys activity (Sessions 3–4), a grid is provided on the screen with dots for every 10 turtle steps. In the Feed the Turtle activity (Session 6), no such grid is provided; however, half of the pilot students drew dots on a paper replica of the computer screen to help them determine an input to the forward command. Also, when the Missing Measures problems were presented (Session 8), Anne strongly objected, "But there's no dots! How do we know how far to make the turtle go?"

The Missing Measures activity helped some students begin to use other superordinate units (units greater than one) informally at first and in a more sophisticated manner in the later problems. For example, David figured out the length of 40 for the one segment by matching the width of his figure to a different segment labeled 40. "It's about forty steps because I measured it with my finger. It's the same as this one." However, on the final interview, David found the missing distances for each figure by drawing dots marking units of 10.

By the time students were engaged in the 200 Steps activity (Session 10), some of the students were still using a figurative partitioning strategy to generate rectangles of various side lengths. For example, Anne usually imposed and used dashed lines to measure her rectangles. This appeared to help her find several solutions to the problem (e.g., sides of 80 and 20; 70 and 30; 60 and 40; 50 and 50); however, a disadvantage was that she was resistant to any length that was not a multiple of 10.

Some students who were initially hesitant to use units other than one, eventually used superordinate units after working in the computer environment. For example, on the Maze Steps game in the field test (Session 2), Luke counted by fives only if there were corners marking each five steps of the path; that is, he needed these marks to count by superordinate units (these are superordinate units to us; as noted below, they may not be conceptually superordinate to the student). Once he saw units of 10 marked off on the computer, he counted by tens. He could work with a superordinate unit, but apparently only did so if such a unit was the smallest perceptual unit available to him. Further, there may have been limits on Luke's unit; it may not have been a unit-of-units, but merely a "countable figural unit of ten" (Steffe & Cobb, 1988).

Some students showed signs of believing that the use of marks was a "crutch." Similar to the pilot students, Monica preferred to place marks down and count them by ones or tens (on Logo tasks), depending on the situation. In a later interview, she made marks with the eraser end of her pencil, but wanted to dust them away before turning her paper in, believing that the marks represented a less sophisticated approach. The performance of all the field-test students improved on the postinterview and only one student made marks. Thus, at that point, most students were operating at a more abstract level. The ability to impose a meaningful number on a line segment without figuratively (e.g., physically or graphically) partitioning the segment represents a significant advance. Students can step back and do anticipatory quantification, applying their number schemes to measurements of a geometric figure. Figurative partitioning may be necessary for the student who has not constructed an abstract conception of length and length measure. We will return to these issues in the conclusions section.

Iterating. Additional support for these conjectures can be inferred from the size of the steps students physically walk and draw. When the students played the role of the student-robot following others' directions (Session 1), they would usually change the size of their actual steps, with the step length decreasing for each successive line segment constituting the figure. Also, when they drew dots on figures, they most often attempted to keep the intervals between the dots equal in length; however, when the path changed direction, they often changed the scale, with the interval between the dots decreasing toward the end of the last line segment. Finally, they also followed this pattern when they drew figures to illustrate the effects of Logo commands. For instance, Barb read the instructions: fd 7, rtf (rtf is a *Geo-Logo* command that stands for "right face" and is equivalent to "right 90°"), fd 7, ltf ("left face"), fd 7, and David played the role of the turtle robot, following these commands. The teacher asked David, "Can you draw a picture of what you walked?" The path he drew had exactly the correct headings, but the lengths decreased for each line segment (even though they were all the same number and he counted to that number while drawing each line segment; see Figure 3). These observations suggest that students have an intuitive sense that the steps should be equal in length, and they tend to maintain this equality when the steps can be created by motions (of the body, hand, or eye) of equal extent at the same orientation. However, once their iterative behavior is interrupted by a turn, they do not have the predisposition to maintain that equal length in new directions. This tendency may arise from beliefs about constraints on the size of units and about reasons that equal lengths are necessary. That is, the students may focus on orientation and counting the steps at each orientation and see no need to coordinate the length of the steps. This phenomenon was pervasive when children were walking paths and was observed frequently in paper-and-pencil environments.

Combining and Decomposing

Combining computer commands. From the beginning of the unit, and increasing throughout, emphasis is given to combining and decomposing measures, especially lengths. The first computer activity, the Get the Toys game (Session 3), achieved its purpose of motivating combinations. The decreasing energy of the battery was especially salient to the students and motivated them to combine old commands to create new, more efficient ones. Early in their play, Anne told her pilot test partner, “You should go back 50. Because thirty plus ten plus ten is fifty. It’s less commands to type one bk 50 than three backwards, 30, 10, and 10.”

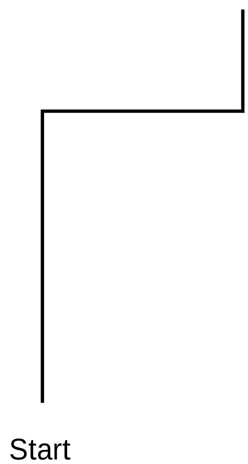


Figure 3. A path drawn by David to represent what the turtle would draw given the commands, fd 7, rtf, fd 7, ltf, fd 7

Students did not always combine commands when possible, however (e.g., these same girls did not combine 80 and 10 when returning). Growth was evident for most students; during their second round of the game, these girls did begin to combine commands before the second command was entered (e.g., “We need twenty more; eighty plus twenty; change fd 80 to fd 100.”).

Oscar and Peter combined commands in a different manner. They appeared to need to use successive approximation, with a series of forward and back movements, to get the turtle located correctly. Only then would they edit the commands to produce a more efficient solution. For example, they once entered fd 80 and when the turtle hit a wall, they started over, entering fd 50 fd 10 fd 10. To save energy, they combined these three commands to fd 70. On other occasions as well, they needed to see all the commands enacted before they grasped what the situation required.

Monica and Nina became adept at combining commands fairly quickly; even away from the computer they could answer correctly questions such as “If you had gone forward forty and back ten, how would you combine the two commands,” justifying that in this case one “needed to minus because the turtle is coming back, it is not going on.” They were also competent at decomposition and reversals in these situations. For example, in Feed the Turtle (Session 6), they had gone forward 60, at which point the turtle had eaten berries. They saw they had farther to go. Monica suggested that they change the command to fd 70. They tried it and found that the turtle went over the berries without eating them. So, they erased the fd 70 and entered fd 60 and, afterwards, fd 10.

We infer from this that Monica and Nina were at a level of reasoning above Oscar and Peter because they could conceptualize the change as part of the new command before they saw it. Furthermore, Anne, Monica, and Nina gave evidence that they had generalized their number and arithmetic schemes to include situations of length and connected line segments (Steffe, 1991). Monica and Nina had also generalized their part-whole schemes, in that they were able to maintain a 70-step length as a whole, while simultaneously maintaining 60- and 10-step lengths as parts of this whole. Oscar and Peter did not appear able to disembed part of a length from the whole in this way. Rather, their solutions were performed by trial-and-error, and their combinations post hoc; indications were that these were combinations of numbers (inputs to *Geo-Logo* commands), rather than lengths (*connected lengths*, which we define, following Steffe [1991], as generalizations of a person's number and arithmetic schemes to include linear quantity, yielding measured line segments that can be subsumed to a part-whole scheme connected to both imagistic and numerical mathematical objects).

Combining lengths. In the first few days of the pilot test, two incidents occurred that indicated the nontrivial nature of Missing Measures problem situations, which were intended to build students' ability to combine lengths. The teacher challenged the students to command a student-robot to walk a closed path to see if she would indeed return to her beginning position (Session 1). The path that they drew is shown in Figure 4.

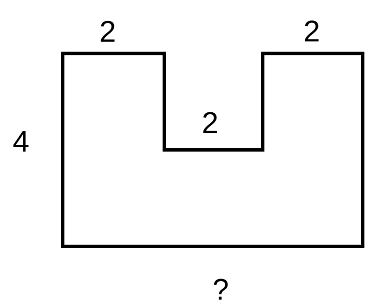


Figure 4. A closed path drawn by children to give to another student playing the role of a turtle robot (It was initially unlabeled.)

They quickly labeled the lengths of several line segments. Even for this path that they had created, however, they had real difficulty determining the length of the bottom line segment, that is, the input to the final forward command. They chose 6—a correct choice, but its genesis was Barb's proposal to add the first and second forward inputs, 4 + 2. No one objected to this.

The other incident occurred earlier the same day, when the teacher indicated the various paths laid out on the floor that the students had walked and asked, "How would I know which one was the longest path?" Barb suggested that "You could measure." Only Charles proposed the notion that one could combine actual physical lengths: "You take this, and you add this one to it. Then add this one (laying each out on the floor, "unbending" each path) and that would be the length." What is pertinent is that, throughout the remainder of the pilot and field test, there is no record of any other student using a real or metaphorical solution that involved "unfolding" a path.

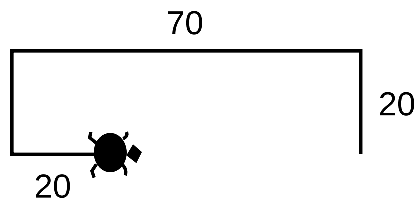


Figure 5. An early Missing Measures problem

What they did do to determine lengths perhaps should not have been surprising. The teacher presented the first problem, illustrated in Figure 5 (Session 8). Recall that Anne protested, "But there's no dots! How do we know how far to make the turtle go?" After the four pilot-test students discussed the procedure, they agreed about the measures of the segments and determined that another forward command was needed. Charles indicated that its input should be 20—"Because the space looks about the same as that side that is 20"—holding two fingers to indicate the length of the side labeled 20 and moving his fingers to the unlabeled space. (This is one part of a complex strategy that will be discussed in more detail in a succeeding section; we call it an imagistic move-segment-numerically adjust strategy.) David said, "No, it's 20, because twenty plus sixty equals eighty (the length of the opposite side)." They tried this solution out at the computer and solved other similar problems. In the group discussion that ended the activity, there were signs that they were assigning numbers to figures as measures of length, abandoning early guesses that they no longer considered viable. For example, explaining his solution to the problem shown in Figure 6b, Charles said, "I thought the missing part on the right side was ten, because it was such a little space. But now I figured it out: 40, 25, so … 65. So, it has to be a 20, not a 10. Because twenty plus the forty-five would be sixty-five."

Students in the field test also used naive strategies initially. All students, however, came to find more useful correspondences in one of two ways. For some, *Geo-Logo*'s feedback alone prompted them to review their choices. Nina, for example, typed fd 50 for the final side of Figure 6a, but after seeing that the figure did not close, mentally calculated the missing length by looking at the labeled measures on the opposite sides. For others, the teacher scaffolded their problem solving, usually by gesturing and saying, for example, "See, if you moved this part over [e.g., the side labeled 40 in Figure 6], and this one too [e.g., the side labeled 25], what would that tell you about the length of this side [gesturing along the right-most side, extending the gesture to the bottom portion of the side]?"

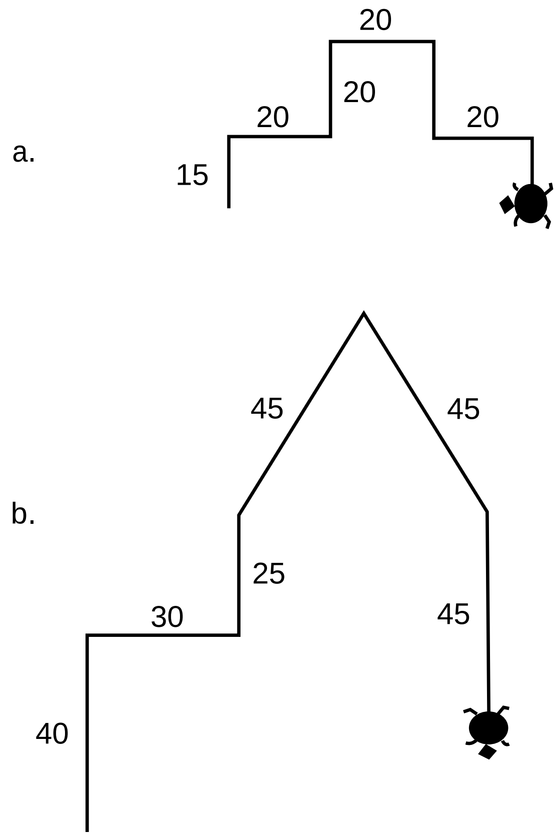


Figure 6. More complex Missing Measures problems

Some took longer than others to develop more sophisticated strategies. For example, on the first Missing Measures problem he attempted, Luke used visual estimation: “Try 50. I don’t know how much for sure.” Even after observing others using different, successful, strategies that he had the ability to appropriate, he would return to guessing. On Figure 5, Luke placed his finger down on a 20-unit line segment and then marked off 3 such units to determine the missing length of the figure. This is consistent with the observation that Luke had not combined commands often in the Get the Toys and Feed the Turtle games (Sessions 3 and 6, respectively), instead preferring to erase commands that he evaluated negatively (“That didn’t work”) and try new numbers. In both situations, Luke did not abstract the mathematical problem to a strictly numerical form, but preferred to take the “safer,” and often more perceptually based, course. In Missing Measures situations for rectangles (Session 8), however, Luke abstracted and then operated on the numbers. For example, given a rectangle to finish (such as in Figure 5, but with one long side of length 100 and an opposite segment of 78), he offered, “fd 22 because seventy-eight and twenty-two is one hundred.” This revealed competence Luke possessed but was not wont to use in more complex figures. To solve rectangle problems, Luke mentally moved one line segment to precisely coincide with the others; he showed the motion with his fingers when asked and used language such as “I dropped that down.” He then assimilated these two lengths into a part-part-whole scheme and immediately used arithmetic procedures to produce a valid answer (the move–segment–part-whole strategy used by David in justifying, “Because twenty plus sixty equals eighty”). For more difficult missing lengths tasks, Luke also used mental motions; however, in these situations he usually moved any convenient labeled segment over the missing length (without precision) and used the former as a unit with which to estimate the measure of the missing line segment—the move-segment-numerically adjust strategy. This constituted a more perceptually based “fall back” strategy for Luke in more complex geometric situations. However, Luke’s competence in simpler situations reveals that he could operate on connected lengths. Other students were observed similarly operating on connected lengths only for rectangles.

What makes the rectangle a less complex situation? There are fewer elements requiring attention; however, another reason is that the line segments are connected to the other pair of sides perpendicularly, not unlike dimension labels used in drafting. The perpendicular segments may also serve as guides for the spatial transformation of mentally sliding one line segment to correspond to the other (the opposite side). In contrast, on the task in Figure 6b, students must bring several segments into alignment and, moreover, may have problems assigning a role to the 30, which, along with the corresponding segment, lies in between the 40 and 25 but adds nothing to their combined length (from our perspective). Some students did not appear to mentally move the line segments for rectangles. Nina solved a missing length task (as in Figure 6a) with three “steps” by saying that she needed to use “this, this, and this,” simultaneously touching or rather pinching between her fingers the line segments that she needed to add together. She showed no physical movements (even with her eyes), possibly making a gestalt judgment as to the corresponding segments.

Connections Between Number and Spatial Schemes

After gaining experience playing the Get the Toys game (Session 3), Anne said, “Try bk 50. Because we put fd 40 in [pointing at the Logo code], but then we had to put in a fd 10 [pointing first to the code, but immediately then to that section of the path], so it’s got to be bk 50 to get back [gesturing back along the section of length 50].” Such behaviors show an integration between the spatial-geometric form of the turtle’s path and the symbolic-numeric representation of the Logo commands that created it. How common was such integration?

Our observations indicate that the answer to this question differs by task and student. Charles, for example, gave signs of simultaneously applying numerical and geometric schemes (e.g., spatial images). On a missing length problem (Session 8), he had entered fd 40 for the missing bottom side, but had come up short. He mentioned that he forgot the “twenty part” at the top.

Teacher: Do you have to go back and do the fd 20 before the fd 40?

Charles: No! I can just add the 20 now. It doesn’t matter. I should have put 60. My command was wrong. ’Cause the 40 only takes care of this part on top.

For the 200 Steps rectangle task (Session 10), Charles offered: “First I wanted

10 and 10, then I couldn't figure it out, so I made it 20 20 80 80."

Teacher: What if you had wanted it 10 and 10?

Charles: Well, then that would have to be … 90 and 90.

Teacher: How did you do that?

Charles: I just added the 10 on to the two long sides … because you can't leave it out.

By this statement, Charles implied that the total perimeter must be maintained; the difference of 10 between the old side length (20) and the one he had initially planned (10) must be maintained, and the only way to do this was to add the length to the other sides. It is significant that Charles deliberately added one measurement to the length of two sides, as opposed to operating (80 + 10) with no referents. At another time, Charles likewise discussed "making these longer, like from 70 to 80, and making the other two sides shorter … change from 30 to 20" and, at other times, bending a fixed length in various ways. Thus, Charles was operating on numbers as symbolic referents for the lengths of line segments.

Not all students gave evidence of such connections in all situations. Barb and Anne initially tried to "get the numbers to add up" to 200, drawing rectangles but without establishing a noticeable connection between the numbers and the figures they drew; for example, they drew a rectangle and labeled the sides "70," "30," "60," "40." In an activity in which they interpreted numbers as inputs to movement commands (e.g., in the Get the Toys game), they were observed maintaining the numeric/spatial connection. However, when they conceptualized a path as static (and moreover when the path was almost incidental to their interpretation of their task) such as in the 200 Steps problem, they gave no signs of establishing, maintaining, or using such a connection. It was more common for students to operate on numbers, rather than on spatial representations of the geometric figures to be created or an integrated numeric/spatial representation, on tasks such as the 200 Steps and Face activities. That is, their behaviors indicated that their intentions were not to make sense of the numbers as measures of length in a setting of constructing certain geometric figures; rather, they were using procedures in an attempt to find numbers that "added up."

There were also some intermediate levels in such integration. Monica and Nina, for example, appeared to determine intuitively which line segments' labels to add, but with no explicit sign that they based this on an analysis of the figures' properties except in the case of simple rectangle completions, for which they used the notion of "opposite sides equal in length." Once they determined which measures to operate on, they applied arithmetic without establishing any further connections between the measures and the spatial form.

In addition, as we observed previously, a subtle alteration in the spatial/geometric figure was sufficient to interrupt or sabotage the numerical process. On the 200 Steps problem, Anne had asked Barb, "What if it were a square?" Barb replied, "No. I tried that already. It didn't work." She showed a drawn figure with each side labeled "25" and marked into five segments; however, she counted the perimeter 5 … to 25 on one side, then 5 to 25 again, then continued up to 75. Then she said, "You can't do it!" The symmetries of the square appeared to confuse her counting in a way that did not occur with rectangles.

There are caveats we must add to such interpretations. The boys from the pilot test, as noted, showed signs of integration on the 200 Steps task. On some occasions, however, it was difficult to assess this integration. Once, they drew a rectangle and labeled it with the dimensions 50 and 90, albeit with the 90 labeling the shorter side. This did not noticeably concern them. This may indicate a disconnection between the numerical and spatial. It might also, however, be a sign of sophistication, depending on whether the boys tried to use the drawing as a veridical representation of a geometric figure or, in some cases at least, as a place holder that they could label (e.g., as some geometry diagrams). There is some indication that the latter was true; the boys wanted a quick way to check their estimation of the perimeter and they moved to proportional drawings immediately thereafter.

In the Face activity (Session 12), in which students planned a drawing consisting of several figures of given perimeters, connections were more complex. Anne, when planning, drew the shapes she wanted, and only then tried to make the numbers fit. She did not ignore scale completely, but neither did she adjust the drawing. Even though she did not draw to scale, she was conscious of figures fitting inside one another and that the numbers thus had to be relatively ordered.

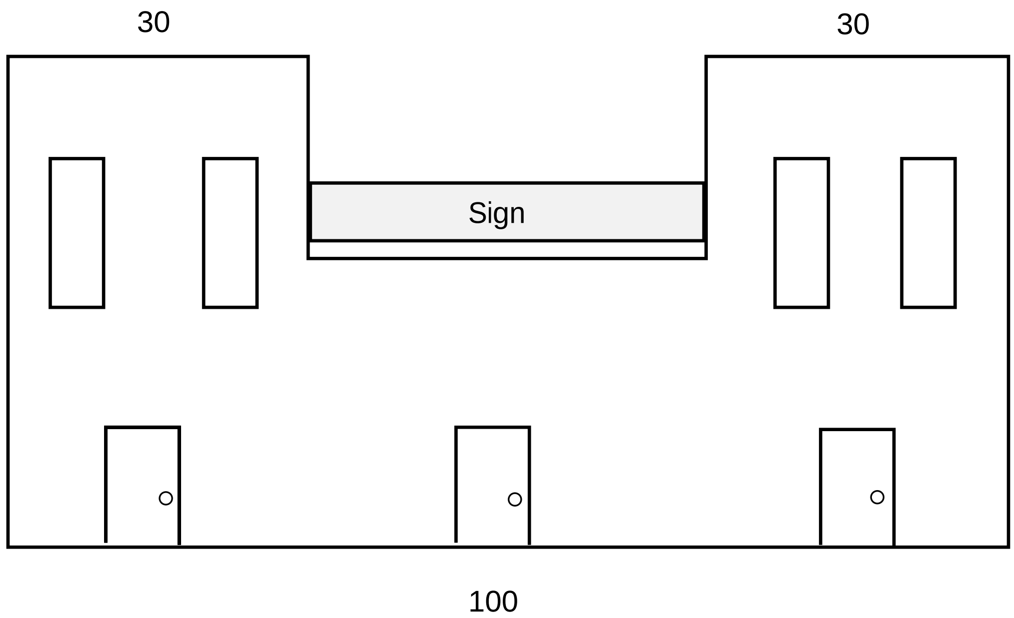
Using Strategies in Different Situations: Interviews and Paper-and-Pencil Tasks

Results on items from the interviews and paper-and-pencil tests provide information on students' performance in other contexts. These contexts differed in that (a) they occurred either before or after (not during) the treatment; (b) they constituted near or far transfer tasks relative to the unit's activities; (c) they were strictly paper-and-pencil, with no support from the use of the computer; and (d) they were administered in a "test" format with no assistance from adults or peers. We wanted to observe students' use of strategies in these new contexts.

The paper-and-pencil test included two Missing Measures problems that assessed strategy use on transfer tasks; differences between performances on these two different tasks were also revealing. The first, a non-Logo near transfer task, is shown in Figure 7a. Anne used a less sophisticated strategy until prompted by an adult. She said, "It's 30. It's the same as this other 30" (a version of the move-segment-numerically adjust strategy). The interviewer asked about the 100. She then added the lengths of 30, 30, and 30. When she noted the result, 90, she tried changing her 30 to 39, added the numbers and, getting 99, stated that the exact answer was 40. Thus, Anne had competence that was demonstrated only in the presence of adult or computer feedback.

For the seven students in the field test who were administered this task on both the pre- and the posttest, three did not evince any change in correctness, two changed from almost correct to correct responses, and two changed from correct to almost correct responses. Where there was credible information on strategy use, all students used visual guessing on the pretest and all but two used it on the posttest. These two both changed from correct responses ("It's more than thirty … forty") to close responses based on informal measurement (e.g., "50" measured with finger widths). Of all 38 students, 10 (26%) gave the correct response on the posttest.

a. Here is a building. The builders want to build a sign.
How wide must the sign be to fit exactly?



b. Write a Logo procedure to draw this figure.

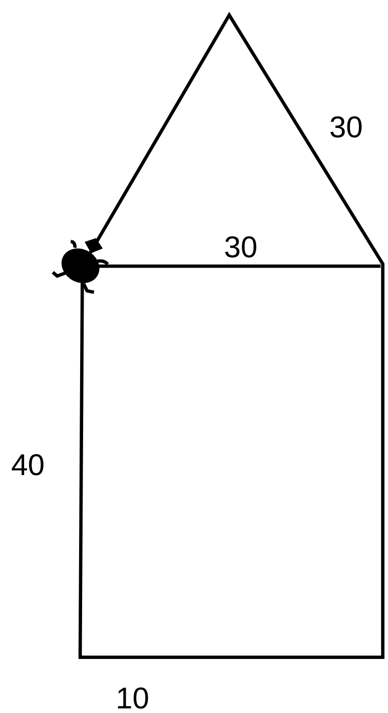


Figure 7. Missing Measures problems from the paper-and-pencil test; a. was Item 6 on that test; b. was Item 7

The second Missing Measures problem was set in a Logo context (Figure 7b). Not surprisingly, given their limited Logo experience, only one of the seven students attempted this item on the pretest; he did not get any commands correct. Performance improved significantly on the posttest. All of these seven students wrote correct commands for the first four forward commands; six wrote correct commands for all other forward commands.

Across all 38 students, for the length commands for which the length was given, 60% were correct; for the length commands for which no length was given (i.e., Missing Measures), 55% were correct. Five students did not attempt the item. Three students got all the commands correct.

Perhaps most interesting is the difference in students' performance between the ostensibly easier missing length problem in Item 6 (Figure 7a) and that in Item 7 (Figure 7b). Students solved missing lengths problems only in the Logo environment. Nonetheless, the support of the real-life situation was expected to aid performance on Item 6. This could be considered merely a problem of generalization. (Support for the flexibility and generality of the missing lengths strategy has since been incorporated into the curriculum, with the presentation of more real-world problems.) Although this traditional "transfer" explanation probably captures some truth, there are alternative explanations. First, the gap between the labeled measures is significant in the "sign" task. Second, the difference also can be attributed to students' evaluation of the situation, including necessary precision. In previous noncomputer measurement experiences, students may have found that informal measurement was adequate. In their Logo experience, precision was necessary.

The last set of relevant paper-and-pencil items was tailored after the 200 Steps and Face activities and thus provides an assessment of students' ability to transfer their strategies to new situation in which they had to generate lengths. The results indicate that the students applied their newly formed strategies and were more likely on the posttest to connect the numbers for measures to the geometric forms they created.

One task asked the students to draw and label the lengths of the sides of a rectangle, square, and equilateral triangle, each with a perimeter of 12. For all three figures, five of the seven students who took both the pretest and the posttest moved from incorrect or no response on the pretest to correct responses on the posttest. The strategies of four students could be identified. One changed from guessing on the pretest to using used trial-and-error with correction on the rectangle, guessing 4 and 3 for the measures of the sides and adjusting to 4 and 2. The other strategies for the rectangle, as well as for the square and triangle, were equally divided between addition (e.g., "I knew that 3 + 3 + 3 + 3 would be 12") and multiplication ("It's got to be 4 for the triangle, because 3 x 4 = 12") strategies. Change in the connection between the numbers and geometric forms the students created could be identified for six students on the rectangle and five on the other two figures; in all cases, they moved to numbers correctly reflecting the geometry of the figures. We interpret this as implying some mutual assimilation of the students' numerical and geometrical schemes, an issue to which we turn in the following section.

For the three figures, about 50% of the students had completely correct perimeters. Most strategies were indecipherable, although about 18% of the students showed addition or multiplication algorithms on their papers. For the rectangle and the square, 58% of the students showed labeled drawings in which the numbers correctly reflected the geometry of the figures, whereas 68% of the students did the same for the triangle.

Two additional tasks similarly revealed that students changed their strategies over the course of the instructional unit. One interview item posed another Missing Measures problem (Figure 8a); data are provided in Tables 2 and 3. This item was more complex than the problems presented in class; the fact that they did not have a computer as a problem-solving tool increased the challenge considerably. Of the 13 students who were interviewed, 9 correctly identified some missing lengths on the postinterview, compared to none on the preinterview. In the preinterview, 7 of these students did not attempt the problem; the other 6 used naive guessing strategies. On the postinterview, all but one student used one or more strategies. As shown in Table 4, for the four unlabeled line segments in this item, only 6 responses (12%) showed no strategy, 13 (25%) responses revealed the use of less sophisticated strategies, including visual estimation, gross comparing, and figurative partitioning, 9 (17%) responses showed use of side lengths but numbers were used incorrectly, and 24 (46%) responses indicated correct application of arithmetic.

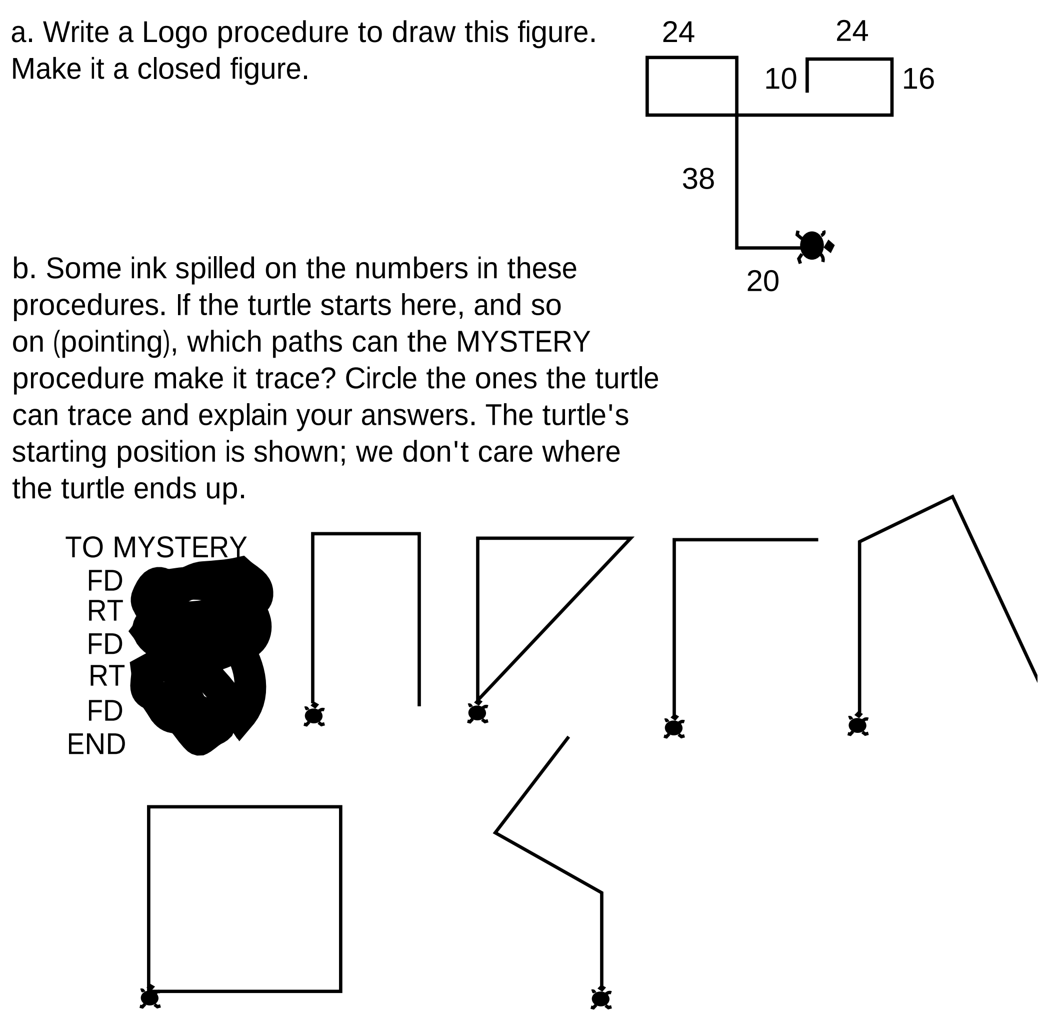


Figure 8. Missing Measures problems from the interview protocol

Table 2
Responses of the 13 Interview Students to the Missing Measures Problem in Figure 8a

Missing lengths	Pretreatment	Posttreatment
No lengths specified correctly	13	4
Lengths correct only if given on figure (i.e., labeled)	0	1
Missing lengths correct only if equal to another given length	0	3
Got computed lengths only if vertical	0	4
Computed lengths correctly, but error in arithmetic calculation	0	1

Table 3
Strategies Used by the 13 Interview Students to Determine the Length of the Four Unlabeled Line Segments in the Problem in Figure 8a

Strategies	Pretreatment	Posttreatment
No response, guessing, random	48	6
Visual estimation, ignored other information on figure	1	9
Used simple measuring/gross comparing (e.g., with pencil, similar to move, segment, numerically adjust strategy)	0	2
Figurative partitioning (e.g., made dashes and counted)	0	2
Used numbers incorrectly		
omitted one or more necessary side lengths	1	4
used geometrically wrong side lengths	0	5
Applied arithmetic correctly		
(move-segment-part-whole strategy)	0	14
but assumed 38 was entire vertical amount	2	9
and combined missing length with a given length in a single fd command (i.e., fd 54 for fd 16 fd 38)	0	1

Note. Total number of responses = 52 for each of pretreatment and posttreatment (13 responses on each of four line segments)

Data from interview Item 3 (Figure 8b) are shown in Table 4. This Logo-based item shows considerable growth from pre- to postinterview, even under the condition that both the response and justification had to be correct (e.g., three argued that there were “too many commands” instead of “too few”). Similarly, there were dramatic changes in the justifications. For example, across all students interviewed and summing across six paths, the number of responses that gave no justification fell from 39 (50%) to 13 (17%), and the number of misinterpretations of Logo code decreased from 7 to 0. Most students gave correct justifications using one-to-one correspondence of commands, either by tracing or by counting; 8 referred to the geometric property of closed/not closed. Thus, students of this age were able to learn to connect symbolic and graphic representations, at least in the realm of Logo code.

Role of Geo-Logo

There are five characteristics of the *Geo-Logo* computer environment that aided construction of units for these students. First was a change in problem situation. The small steps (and thus larger numbers) on the computer, compared to the students’ actual steps or the steps in the Maze Paths game (Session 2), lead to conceptualizing and counting superordinate units (it is difficult even to image a single turtle step). So, work in this environment may encourage the creation of a more elaborated measurement scheme. Conversely, this characteristic could hinder development for students who are younger or have less developed concepts. We did not observe this problem in this study.

Table 4
Responses to Interview Item 3, Shown in Figure 8b

Response	Pretreatment	Posttreatment
No response	24	3
Incorrect or correct response, but wrong justification	27	16
Claimed impossible incorrectly, but had cogent rationale	10	15
Completely correct	17	44
Justification of responses		
No justification	39	13
One–to–one correspondence (or lack of) between path components and commands, indicated by tracing	5	19
One–to–one correspondence (or lack of) between path components and commands, indicated by counting or by other quantification method	20	31
Lack of a particular required command (e.g., “there is no lt in the procedure”)	7	7
Refers to closed/not closed	0	8
Misinterprets Logo commands (e.g., turns can be 90° only)	7	0

Note. Total number of responses = 78 for each of pretreatment and posttreatment (13 responses on each of four line segments).

Second, the computer provides feedback that students can use to reflect on their thinking. There are numerous instances in this study that support this claim (Schwartz, 1989; Weir, 1987). As one example, recall Anne and Barb’s “rectangle” with sides labeled “70,” “30,” “60,” and “40.” Anne used dashed lines to keep track of the “10-length segments” of each side; however, noncongruency of these small measurement segments allowed the drawn figure to be rectangular. When she tried this solution on the computer, however, the resulting open path immediately led her to relate her side lengths to the properties of a rectangle. Anne exclaimed, “It’s not a rectangle!” Barb added, “Because it’s like this [draws in the air]. Because one side was not as long as the other one. So, we have to do that again.”

The girls changed their procedure to a rectangle of sides of 60 and 30 and believed at first that this was 200 steps in total length. With further discussion after a prompt by the teacher, they changed to a rectangle with sides of 70 and 30. In a subsequent discussion at the field test, the teacher asked students if these commands would produce a rectangle: fd 40 rt 90 fd 65 rt 90 fd 20 fd 20 rt 90 fd 50 fd 15 rt 90. Both Monica and Nina agreed that it would be a rectangle until they tried drawing it on paper. Monica’s drawing was not a rectangle; Nina’s was. They still disagreed, even after Nina explained to Monica that “twenty and twenty added up to forty, which is actually equal to the other forty” and that the teacher “was only tricking” them. Monica felt the need to check this on the computer but then was convinced; she explained the 50 + 15 =

65 to other students. The class ended with Nina and Monica giving each other “tricky” lengths. This illustrates not only the role of computer feedback but also our assertion that Monica and Nina possessed schemes for connected numbers.

Geo-Logo’s feedback also was important to students’ independent problem-solving efforts, as it was for Nina in her first attempts to solve Missing Measure problems. On a different problem, Charles found and corrected his error after entering fd 40, which he determined was the missing length when it was actually only one component of that length.

Third, the computer context was motivating for these students. Oscar and Peter were, in the teacher’s opinion, the “lowest ability” students in mathematics. With paper and pencil, they often solved the problems with visually based guessing, but on the computer, they were convinced that only the deliberate use of arithmetic was adequate. They needed pencil and paper for almost all the calculations (e.g., as an afterthought in the Get the Toys game, they did not use such strategies in Missing Measures problems). They were, however, highly motivated by this computer activity, and their teacher had to stop them from fighting between themselves as to who would get to do the arithmetic for each figure.

The flexibility and dynamic connections between symbol and graphic may have combined to engender a fourth advantage. Anne and Barb never combined their commands on the Get the Toys game when they wrote Logo commands by hand, but they did combine them when they typed the commands into the computer. Only later did they begin to combine commands on paper as well.

A fifth advantage of the *Geo-Logo* environment relates to the integration of the spatial and numeric. We have commented that this occurred rarely, especially in noncomputer situations. When it did occur, the statement was often not about spatial extent, or length, but about dynamic movement. For example, Nina and Monica used arithmetic to solve problems, but their language was about the turtle: “It’s going on 50, then 30 more, so that’s….” The emphasis here is not on the geometric figure as much as it is on the turtle’s movements. Thus, the emphasis on physical action and the dynamic connections between the symbolic and graphic representations in *Geo-Logo* facilitated students’ development of such connections for themselves. This conclusion must be tempered with a recognition that these connections were tenuous and situation-bound in many instances. It is significant, however, that we never observed students using a point-counting process (counting points rather than line segments) to define units (Cannon, 1992); rather, they imaged and counted line-segment units.

CONCLUSIONS

We observed three levels of strategies for solving our different length problems, and we hypothesize that students pass through these levels during their development of length concepts. Students operating at the first level did not segment lengths and also did not connect the number for the measure with the length of the line segment. Rather, they applied general strategies such as visual guessing of measures and naive guessing of numbers or arithmetic operations (that were sometimes disconnected from the spatial structure of the problem, as with Missing Measures situations in which measures of orthogonal segments were added to yield the measure of a side parallel to one of these segments, as in Figure 4). These students tended to be those identified by their teacher as being low in mathematical ability. Oscar and Peter could be so categorized. We never observed them making statements that would indicate that they were operating on quantities. Furthermore, when they were asked to draw a figure such as a rectangle with certain dimensions, there was no discernible use of those dimensions in their drawings. They interpreted the problems as numerical problems, rather than as measurement problems. However, they did make sense of the problems in their own way, rather than merely trying to discern what answer the teacher wanted; although some students were observed using memorized procedures such as addition algorithms in situations in which other strategies such as mental computation seemed to adult observers to be much less tedious, the students were nonetheless solving a (numerical) problem that was meaningful to them.

Strategies at the second level were most common among these third graders. They drew hash marks, dots, or line segments to partition or segment (i.e., not maintaining equal length parts) lengths. A turtle step is a small unit (1 mm or less on most monitors); moreover, students' experience is probably such that objects 100 units in length are substantial in size. These factors may have made the abstraction of the turtle step difficult for students who wished to assign numbers in a meaningful, quantitative manner. Therefore, they marked off lengths in units that made sense to them, usually units of 10. They needed to have perceptible units such as these to quantify the length (Steffe, 1991). This led to interesting use of numerical units of units. It may also, however, explain some students' resistance to making 200-step rectangles whose side lengths were not multiples of 10, in that 10 may have been a nondecomposable unit for them in that context, at least before interactions with other peers and the teacher produced a perturbation.

Students using strategies at the third level, like those of the first level, did not use figurative partitioning (or ceased using partitioning at some point). However, they did use quantitative concepts in discussing the problems, drew proportional figures, and sighted along line segments to assign them a length measure. They used a move-segment-part-whole strategy on most Missing Measures tasks. Therefore, we assume they had interiorized units of length and had developed a measurement sense that they could impose mentally onto figures. These observations substantiate Steffe's argument that these students have created an abstract unit of length (Steffe, 1991). This is not a static image, but rather an interiorization of the process of moving (visually or physically) along an object, segmenting it, and counting the segments. When consecutive units are considered a unitary object, the student has constructed a "conceptual ruler" that can be projected onto unsegmented objects (Steffe, 1991). As components of an automatic, anticipatory mental structure, such processes are used without conscious awareness of them by these more sophisticated students and adults.

We hypothesize that once students at the first level have had sufficient physical measurement experience iterating and partitioning into units, they construct schemes that allow them to partition unsegmented lengths. Such second-level schemes are figurative; that is, they need to use physical action to create perceptual partitions. In solving problems, these partitioning schemes develop to include the constraint that equal intervals must be maintained. This constraint leads to the construction of an anticipatory scheme, because the equal-interval constraint can be realized most efficiently when it is done in imagery, in anticipation, without forcing perceptual markings. At this point, third-level strategies emerge.

Finally, students' individual predispositions and capabilities in connecting numeric and spatial schemes affected their choice and employment of these strategies. Some students, such as Luke, were skilled with numbers and computation, including mental computation. Luke, however, gave few indications of performing operations on spatial representations, and his computations did not appear to be linked to the quantity in the situation—length, in this case. This should not be construed as implying that Luke used no imagery in performing arithmetic, but only that when doing exact computations he did not use imagery of the length of the segments constituting the geometric shapes in the situation we had created. In more complex situations, he used mental computations that were connected to these lengths, as he iterated another length to estimate the missing length; however, he did not consider other relationships within the geometric figure.

Monica, on the other hand, is representative of students who do connect, at least in some situations, their knowledge of numbers and measurement quantities. Monica used arithmetic to solve all Missing Measures tasks. Luke preferred to use visually based guesses when figures were more complex, even though his arithmetic skills per se were as sophisticated as Monica's, if not more so. Thus, students who have connected numeric and spatial representations may evince different problem-solving strategies in geometric situations from those who have forged fewer such connections.

IMPLICATIONS FOR INSTRUCTION

As teachers, we might view students who did not connect spatial and numerical schemes, and used only the latter to solve problems, as having different but equally effective solution strategies. We argue, however, that such students would benefit from activities that guide them to synthesize these two schemes. First, students need to pay attention to the scale that is provided for certain figures (e.g., in work with geometry, maps, or graphs.) Second, this study's data indicate that those students with connected schemes had more powerful and flexible solution strategies at their disposal for solving spatial problems. Third, and in a similar vein, the activity of connecting mathematical ideas is in itself a valuable mathematical activity.

Students at the first and third levels of solving length problems did not use hash marks to segment or partition line segments. Teachers should take special care to observe these students' interpretations of the task, for they need to engage in quite different types of activities. Students at the third level may be challenged with more difficult missing lengths tasks. Students at the first level need to engage in partitioning and iterating lengths, continually tying the results of that activity to their counting schemes. Tasks in which applying only numerical schemes is ineffective may be especially useful. This is why we changed several missing lengths problems. For example, in an early version of the task shown in Figure 6a, all segments were 20 units; in a later version, two vertical segments were changed to 15 units, as shown in Figure 6a. With the latter figure, students who chose the 15 length segments were shown, through computer work, that their solutions were not adequate. In the following paragraphs, we discuss other implications that are more general and less directly connected to the data but, nevertheless, are warranted.

1. Encourage connections throughout the mathematics curriculum. It appears that for many students, connections between geometric forms and numerical ideas are tenuous at best, even in situations designed to emphasize and develop these connections. Such lack of linkages would appear to limit the growth of number sense, geometric knowledge, and problem-solving ability. Studying more geometry, especially of the type described in the unit employed in this study, may ameliorate this situation.

2. Emphasize students' creation of unit of measure. Most students had to create figural entities to assign a measure to a length that was not partitioned. Activities that emphasize such creation in the solution of meaningful problems are probably important at this (third) and other grade levels, and may have such ancillary benefits as more elaborate and flexible use of representational tools such as the number line.

3. Employ nontrivial problems. Activities such as the computer games Missing Measures, Equilateral Triangle, and 200 Steps tasks were nontrivial problems for these third-grade students. They were also engaging, however, and rich in opportunities for exploring, using, and communicating about mathematics.

4. Choose meaningful tasks that encourage mathematical analysis. Observations confirm previous findings that many students of this age initially rely on visual cues only and eschew analytical work such as using dynamic imagery and arithmetic to find precise mathematical and programming relations within the geometry of the figure (Clements & Battista, 1992; Hillel & Kieran, 1988). There is little reason for students to abandon limited surface-level visual approaches unless they are presented with tasks whose resolution requires an analytical approach, as did most of the tasks in the Turtle Paths unit.

5. Use computer environments such as *Geo-Logo* that encourage and support the use of mathematical concepts and processes. The *Geo-Logo* environment was critical in providing meaningful tasks. Students' integration of number and geometry was especially potent and synergistic in the *Geo-Logo* environment. The geometric setting provided both motivations and models for thinking about number and arithmetic operations. The motivations included game settings and the desire to create geometric forms. The models included length and rotation as settings for building a strong sense of both numbers and operations on numbers, with measuring and labeling tools supporting such construction. Conversely, the numerical aspects of the measures provided a context in which students had to attend to certain properties of geometric forms. The measures made such properties (e.g., opposite

sides equal) more concrete and meaningful to the students. In addition, the change in problem situation encouraged the use of superordinate units. The dynamic links between these two domains structured in the *Geo-Logo* environment (e.g., a change in code automatically reflected in a corresponding change in the geometric figure) facilitated students' construction of connections between their own number and spatial schemes. Finally, *Geo-Logo* provided feedback that students used to reflect on their own thinking.

REFERENCES

Campbell, P. F. (1987). *Measuring distance: Children's use of number and unit*. Final report submitted to the National Institute of Mental Health Under the ADAMHA Small Grant Award Program. Grant No. MSMA 1 R03 MH423435-01. University of Maryland, College Park.

Cannon, P. L. (1992). Middle grade students' representations of linear units. In W. Geeslin & K. Graham (Eds.), *Proceedings of the Sixteenth Psychology of Mathematics Education Conference: Vol. I* (pp. 105–112). Durham, NH: Program Committee of the 16th Psychology of Mathematics Conference.

Carpenter, T. P., & Lewis, R. (1976). The development of the concept of a standard unit of measure in young children. *Journal for Research in Mathematics Education*, *7*, 53–58.

Clements, D. H., & Battista, M. T. (1989). Learning of geometric concepts in a Logo environment. *Journal for Research in Mathematics Education*, *20*, 450–467.

Clements, D. H., & Battista, M. T. (1992). Geometry and spatial reasoning. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 420–464). New York: Macmillan.

Clements, D. H., & Meredith, J. S. (1993). Research on Logo: Effects and efficacy. *Journal of Computing in Childhood Education*, *4*, 263–290.

Clements, D. H., & Meredith, J. S. (1994). *Turtle math* [Computer program]. Montreal, Quebec: Logo Computer Systems, Inc.

Clements, D. H., & Sarama, J. (1995). Design of a Logo environment for elementary geometry. *Journal of Mathematical Behavior*, *14*, 381–398.

Hiebert, J. (1981). Cognitive development and learning linear measurement. *Journal for Research in Mathematics Education*, *12*, 197–211.

Hillel, J., & Kieran, C. (1988). Schemas used by 12-year-olds in solving selected turtle geometry tasks. *Recherches en Didactique des Mathématiques*, *8/1.2*, 61–103.

Kouba, V. L., Brown, C. A., Carpenter, T. P., Lindquist, M. M., Silver, E. A., & Swafford, J. O. (1988). Results of the fourth NAEP assessment of mathematics: Measurement, geometry, data interpretation, attitudes, and other topics. *Arithmetic Teacher*, *33*(9), 10–16.

Lindquist, M. M., & Kouba, V. L. (1989). Geometry. In M. M. Lindquist (Ed.), Results from the Fourth Mathematics Assessment of the National Assessment of Educational Progress (pp. 44–54). Reston, VA: National Council of Teachers of Mathematics.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author.

Noss, R. (1987). Children's learning of geometrical concepts through Logo. *Journal for Research in Mathematics Education*, *18*, 343–362.

Petitto, A. L. (1990). Development of numberline and measurement concepts. *Cognition and Instruction*, *7*, 55–78.

Piaget, J., & Inhelder, B. (1967). *The child's conception of space*. New York: W. W. Norton.

Schwartz, J. L. (1989). Intellectual mirrors: A step in the direction of making schools knowledge-making places. *Harvard Educational Review*, *59*(1), 51–61.

Steffe, L. P. (1991). Operations that generate quantity. *Learning and Individual Differences*, *3*, 61–82.

Steffe, L. P., & Cobb, P. (1988). *Construction of arithmetical meanings and strategies*. New York: Springer-Verlag.

Strauss, A., & Corbin, J. (1990). *Basics of qualitative research: Grounded theory procedures and techniques*. Newbury Park, CA: Sage.

van Hiele, P. M. (1986). *Structure and insight*. Orlando, FL: Academic Press.
Weir, S. (1987). *Cultivating minds: A Logo casebook*. New York: Harper & Row.

Authors

Douglas H. Clements, Professor, Department of Learning and Instruction, State University of New York at Buffalo, 593 Baldy Hall, Buffalo, NY 14260

Michael T. Battista, Professor, Kent State University, 404 White Hall, Kent, OH 44242

Julie Sarama, Assistant Professor, College of Education, Teacher Education Division, Wayne State University, Detroit, MI 48202

Sudha Swaminathan, Assistant Professor, Eastern Connecticut State University, 83 Windham St., Willimantic, CT 06226

Sue McMillen, Associate Professor, Mathematics and Natural Sciences, D'Youville College, 320 Porter Ave., Buffalo, NY 14201